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LXXXV. *The Magneto-Optical Dispersion of Organic Liquids in the Ultra-violet Region of the Spectrum.*—Part VI. *The Magneto-Optical Dispersion of Acetic Acid and Normal Propyl Acetate. By* W. J. Lewis, *M.Sc., and* G. E. Jones, *B.Sc., Physics Department, University College of Swansea* *.

THE apparatus and experimental method employed in these investigations have been described in detail in Part I.†, and, consequently, only a brief reference to these points is necessary.

The light from a tungsten arc falls on a Bellingham-Stanley polarizing unit designed for work in the ultra-violet, and then traverses the liquid contained in a fused quartz polarimeter tube, which is placed symmetrically inside the core of a solenoid. The polarized beam, after emerging from the liquid, passes through the analyser and a quartz fluorite lens, which brings it to a focus on the slit of a quartz spectrograph. As the beam of light emerging from the polarizer consists of two semi-circular fields with a horizontal line of demarcation, the vibrations in one of the fields being polarized at a slight

* Communicated by Prof. E. J. Evans, D.Sc.

† Stephens and Evans, Phil. Mag. x. p. 759 (1930).

angle with those in the other, two spectra (one immediately above the other) are obtained at the camera end of the quartz spectrograph.

The analyser is set at zero by adjusting its position so that these two spectra have the same intensity throughout their length when the magnetic field of the solenoid is not excited. There are four positions of the analyser for which this is possible ; but one of the two positions of minimum intensity is chosen, as the polarimeter is then in its most sensitive position. The analyser is then rotated through an angle θ, and a photograph taken with the magnetic field excited. A second photograph is also taken on the same plate, corresponding to a rotation θ on the opposite side of the zero with the magnetic field reversed. An examination of the photographic plate shows that there is a line of definite wave-length, which has the same intensity in the upper and lower half of each spectrum. Let λ be the mean wave-length of this line, as determined from the photograph, and θ_1 the rotation at this wave-length due to the quartz ends of the polarimeter tube : then the value of Verdet's constant δ of the liquid for wave-length λ is given by the equation

$$\theta-\theta_1=\delta . \Sigma \mathrm{H} . l,$$

where the summation is taken over the length of the liquid column. The current passing through the solenoid was kept constant at two amperes, and the magnetic field at different points along the axis of the solenoid had been accurately measured. The value of $\Sigma \mathrm{H} . l$ was 12520 ± 20 cm. gauss.

The refractive indices of the two liquids in the visible region of the spectrum were determined by means of an accurate spectrometer in the usual manner, and in the ultra-violet region of the spectrum photographically by means of the quartz spectrograph in which the Cornu prism had been replaced by a hollow glass prism closed by plates of optically worked fused quartz.

The results obtained by the two methods overlapped over a region of the visible spectrum. The mode of procedure in the photographic determination was as follows :—

Three slightly overlapping copper spectra were photographed with the hollow prism filled with (a) the standard

liquid, (*b*) the liquid under investigation, and (*c*) the standard liquid. The standard liquids employed were ethyl and normal propyl alcohols, whose refractive indices in the visible and ultra-violet regions of the spectrum had been measured by Victor Henri *. The refractive index of the liquid under investigation for a wave-length λ_2 can be determined by identifying, in the spectrum produced by the standard liquid, a line of wave-length λ_1, which coincides in position with the line of wave-length λ_2 in the spectrum of the liquid under investigation. Then the refractive index of the liquid for a wave-length λ_2 equals the refractive index of the standard liquid for the wave-length λ_1. The temperatures of the liquids were determined immediately after each exposure by means of a small calibrated mercury thermometer, and the slight shifts in the positions of the spectra were due to small temperature changes. The values of the temperature coefficients of refractive index of each liquid were found by taking photographs of two slightly overlapping copper spectra by means of the hollow prism containing the particular liquid at two known temperatures. From the resulting displacements in the positions of the spectrum lines the temperature coefficients corresponding to different regions of the spectrum could be determined. From a knowledge of the temperature coefficients the refractive indices of the liquids at temperatures differing by a few degrees from those at which they were determined could be calculated.

The liquids employed in this investigation were obtained from Harrington Bros., Ltd., London, and each was subjected to a process of fractional distillation before determinations were made of the refractive indices and the magneto-optical rotations.

The experimental results have been examined in relation to Larmor's † theory of magneto-optical rotation. According to this theory Verdet's constant δ is given by the expression

$$\delta = \frac{e}{2mC^2} \cdot \lambda \cdot \frac{dn}{d\lambda}, \qquad (1)$$

where n is the refractive index, $\frac{e}{m}$ is the ratio of the charge

* 'Études de Photochemie,' Victor Henri, p. 61.

† Larmor, 'Æther and Matter,' Appendix F.

to the mass of the resonators, and C the velocity of light. In the above expression the charge e is measured in electrostatic units and the magnetic field in electromagnetic units.

If the magneto-optical dispersion of each of the liquids in the region of the spectrum investigated is controlled by one absorption band in the ultra-violet, and the ordinary dispersion by an equation of the Ketteler-Helmholtz type, it can be shown that

$$\phi = n\delta\lambda^2 = \mathrm{K}\left(\frac{\lambda^2}{\lambda^2-\lambda_1^2}\right)^2, \quad . \quad . \quad . \quad . \quad (2)$$

where K is a constant and λ_1 the wave-length of the absorption band in the ultra-violet.

In addition, it can be shown by combining equations (1) and (2) and the ordinary dispersion equation

$$n^2-1=b_0+\frac{b_1}{\lambda^2-\lambda_1^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that the value of $\frac{e}{m}$ is given by $-\frac{2\mathrm{KC}^2}{b_1}$.

Experimental Results.

Ordinary Dispersion.

Table I. (*a*) *.

Spectrometer Determinations in the Visible.

Acetic acid at 19·5° C.		Propyl acetate at 13° C.	
Wave-length (microns).	Refractive index.	Wave-length (microns).	Refractive index.
·6678	1·3698	·6678	1·3850
·6563	$1{\cdot}3700_9$	·5876	1·3873
·5893	$1{\cdot}3720_3$	·5016	1·3909
·5016	$1{\cdot}3758_7$	·4922	1·3915
·4861	$1{\cdot}3767_5$	·4713	1·3928
·4472	$1{\cdot}3795_6$	·4472	1·3946
·4341	$1{\cdot}3806_9$		

* The refractive indices of both liquids were determined in the neighbourhood of 20° C. The temperature coefficient of propyl acetate was taken as ·00041 per degree C.

TABLE I. (*b*).

Photographic Determinations.

Acetic acid at 19·5 ° C.		Propyl acetate at 13° C.	
Wave-length (microns).	Refractive index.	Wave-length (microns).	Refractive index.
·4536	$1·3790_4$	·5218	1·3898
·4378	$1·3803_9$	·4653	1·3932
·4275	$1·3812_6$	·4590	1·3936
·4026	$1·3838_6$	·4375	1·3954
·3860	1·3859	·4023	1·3988
·3654	$1·3888_3$	·3849	1·4009
·3482	$1·3917_5$	·3700	1·4030
·3466	$1·3920_6$	·3530	1·4057
·3363	1·3941	·3470	1·4067
·3274	$1·3960_3$	·3274	1·4108
·3194	1·3979	·3168	1·4134
·2961	$1·4044_3$	·3065	1·4161
·2883	$1·4071_5$	·3036	1·4169
·2824	1·4093	·2961	$1·4192_5$
·2766	$1·4115_9$		

Fig. 1.

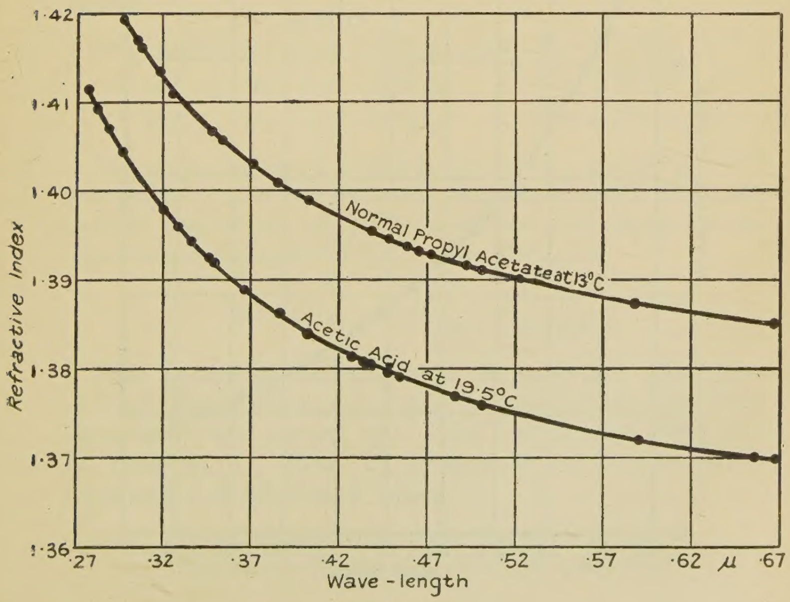
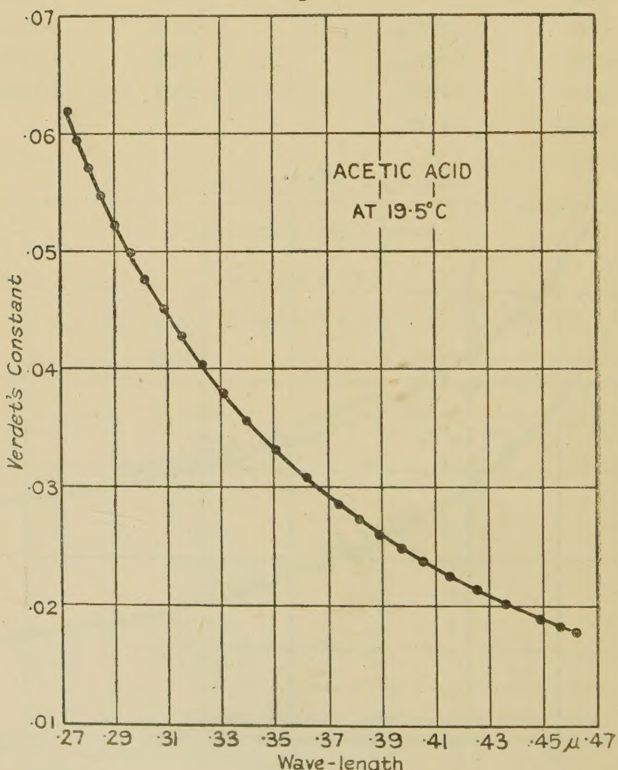


TABLE II. (*a*).

Temperature in ° C.	Wave-length in microns.	Verdet's constant (min./cm. gauss).	Temperature in ° C.	Wave-length in microns.	Verdet's constant (min./cm. gauss).
19·4	·4617	$\cdot0178_1$	19·7	·3396	$\cdot0356_3$
19·4	·4557	$\cdot0183_7$	19·5	·3308	$\cdot0380_1$
19·5	·4483	$\cdot0190_0$	19·5	·3227	$\cdot0403_8$
19·5	·4359	$\cdot0201_8$	19·5	·3153	$\cdot0427_5$
19·5	·4250	$\cdot0213_7$	19·6	·3081	$\cdot0451_3$
19·5	·4148	$\cdot0225_6$	19·6	·3018	$\cdot0475_1$
19·6	·4054	$\cdot0237_5$	19·4	·2961	$\cdot0498_8$
19·5	·3968	$\cdot0249_4$	19·5	·2906	$\cdot0522_5$
19·5	·3887	$\cdot0261_2$	19·5	·2856	$\cdot0546_3$
19·5	·3810	$\cdot0273_1$	19·5	·2811	$\cdot0571_7$
19·5	·3734	$\cdot0286_6$	19·4	·2765	$\cdot0593_8$
19·6	·3616	$\cdot0308_7$	19·4	·2724	$\cdot0617_6$
19·5	·3502	$\cdot0332_5$			

Fig. 2.



Magneto-optical Dispersion.

The specimen of acetic acid employed for the magneto-optical experiments distilled over between 117·2° C. and 117·6° C. at a pressure of 749 mm. of mercury. Its normal boiling-point is 118·1° C. according to the International Critical Tables * and 117·88° C. according to Bousfield and Lowry †. The experimental results are given in Table II. (*a*) and fig. 2.

In Table II. (*b*), values of Verdet's constant and the refractive indices corresponding to certain wave-lengths are given. These were used to calculate the constants of the equation

$$\phi = n\delta\lambda^2 = \mathrm{K}\left(\frac{\lambda^2}{\lambda^2-\lambda_1{}^2}\right)^2.$$

Table II. (*b*).

	λ (microns).	δ (min./cm. gauss).	*n*.
(*a*)	·4483	$\cdot 0190_0$	$1\cdot 3794_3$
(*b*)	·3502	$\cdot 0332_5$	1·3915
(*c*)	·2765	$\cdot 0593_8$	$1\cdot 4115_9$

The following pairs were used to evaluate the constants :—

From (*a*) and (*b*) $\lambda_1 = \cdot 1042_3\,\mu$ and $\mathrm{K} = 4\cdot 713_6 \times 10^{-3}$;
,, (*b*) ,, (*c*) $\lambda_1 = \cdot 1039_5\,\mu$,, $\mathrm{K} = 4\cdot 714_0 \times 10^{-3}$;
,, (*a*) ,, (*c*) $\lambda_1 = \cdot 1043_9\,\mu$,, $\mathrm{K} = 4\cdot 717_3 \times 10^{-3}$.

The mean values of λ_1 and K are $\cdot 1042\,\mu$ and $4\cdot 71_5 \times 10^{-3}$ respectively, and the equation representing the magneto-optical dispersion of acetic acid for the range of the spectrum investigated (·4617 μ to ·2724 μ) is

$$n\delta = 4\cdot 71_5 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2-(\cdot 1042)^2\}^2}.$$

This equation was employed to calculate the values of δ for other wave-lengths, where experimental determinations had been carried out. The values of *n* were read off from fig. 1. Table II. (*c*) gives a comparison of the observed and calculated values.

* International Critical Tables, i. p. 179.
† Bousfield and Lowry, J. C. S. xcix. p. 1432 (1911).

The value of Verdet's constant for sodium light was calculated from the equation, and found to be ·01053 at 19·5° C. According to Perkin * the value of the specific rotation of acetic acid at 18·3° C. is ·8001, and at 20·5° C. ·8039. Taking the value ·8021 to correspond to 19·5° C., the value of Verdet's constant of acetic acid at that temperature (δ for water being taken as ·0131) is found to be ·01051. It is difficult to make an accurate comparison owing to the irregular variations in Perkin's values.

TABLE II. (*c*).

λ.	δ (observed).	δ (calculated).
·4617	$\cdot 0178_1$	$\cdot 0178_2$
·4359	$\cdot 0201_8$	$\cdot 0202_2$
·4250	$\cdot 0213_7$	$\cdot 0213_9$
·4054	$\cdot 0237_5$	$\cdot 0237_7$
·3734	$\cdot 0286_6$	$\cdot 0286_6$
·3308	$\cdot 0380_1$	$\cdot 0380_6$
·3153	$\cdot 0427_5$	$\cdot 0427_3$
·3018	$\cdot 0475_1$	$\cdot 0475_5$
·2856	$\cdot 0546_3$	$\cdot 0546_4$

TABLE II. (*d*).

	Magnetic rotary power relative to that at ·5461μ.	
Wave-length.	Lowry & Dickson.	Present results.
·6708	·648	·648
·6438	·698	·706
·5893	·851	·851
·5086	1·163	1·165
·4800	1·331	1·320
·4359	1·631	1·631

Lowry and Dickson † determined the magnetic rotations of acetic acid for a number of wave-lengths in the visible spectrum, and compared the rotary powers at various wave-lengths with that at λ ·5461 μ. A comparison of their results with those calculated from the magneto-optical dispersion equation, which represents the present results, is given in Table II. (*d*).

* Perkin, Journ. Chem. Soc. xlv. p. 431 (1884).
† Lowry and Dickson, Journ. Chem. Soc. ciii. p. 1067 (1913).

Normal Propyl Acetate.

The liquid was subjected to a process of fractional distillation, and that portion which distilled over between 101·6° C. and 102° C. at a pressure of 76·6 cm. of mercury was retained for the determination of the magneto-optical dispersion. The normal boiling-point of the liquid is given as 101·6° C.* The values of Verdet's constant at various wave-lengths are given in Table III. (*a*), and plotted in fig. 3.

TABLE III. (*a*).

Verdet's Constant of Normal Propyl Acetate at 13° C.

Wave-length (microns).	Verdet's const. (min./cm. gauss).	Wave-length. (microns).	Verdet's const. (min./cm. gauss).
·4621	$\cdot 0192_3$	·3650	$\cdot 0327_9$
·4550	$\cdot 0199_1$	·3609	$\cdot 0336_5$
·4430	$\cdot 0210_7$	·3548	$\cdot 0350_7$
·4384	$\cdot 0216_3$	·3509	$\cdot 0360_6$
·4332	$\cdot 0222_3$	·3450	$\cdot 0374_9$
·4236	$\cdot 0233_7$	·3413	$\cdot 0384_7$
·4182	$\cdot 0240_3$	·3364	$\cdot 0398_5$
·4142	$\cdot 0245_8$	·3328	$\cdot 0408_7$
·4060	$\cdot 0257_3$	·3282	$\cdot 0422_4$
·4013	$\cdot 0264_3$	·3250	$\cdot 0432_7$
·3980	$\cdot 0269_1$	·3212	$\cdot 0445_5$
·3908	$\cdot 0281_0$	·3180	$\cdot 0456_8$
·3860	$\cdot 0288_4$	·3146	$\cdot 0469_4$
·3835	$\cdot 0292_6$	·3114	$\cdot 0480_8$
·3772	$\cdot 0304_2$	·3056	$\cdot 0504_8$
·3727	$\cdot 0312_5$		

In Table III. (*b*), are collected values of the refractive index and Verdet's constant of normal propyl acetate at various wave-lengths. These values, which were taken from fig. 1 and Table III. (*a*) were employed to determine the constants K and λ_1 of the magneto-optical dispersion equation

$$\phi = n\delta\lambda^2 = K\left(\frac{\lambda^2}{\lambda^2 - \lambda_1^2}\right)^2.$$

* International Critical Tables, vol. i. p. 192.

Fig. 3.

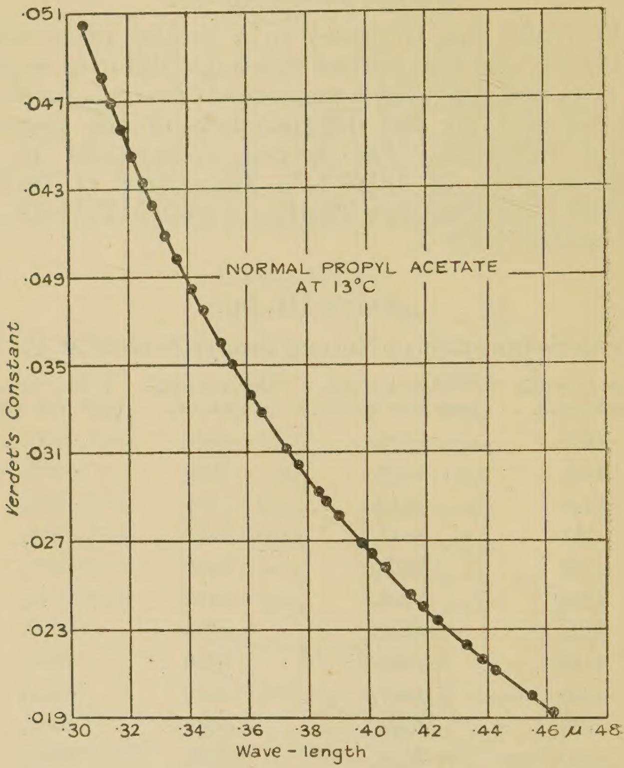


TABLE III. (*b*).

	λ.	δ.	n.
(*a*)	·4550	·0199	1·3939
(*b*)	·4384	$\cdot 0216_3$	1·3953
(*c*)	·4332	$\cdot 0222_3$	1·3958
(*d*)	·4236	$\cdot 0233_7$	1·3968
(*e*)	·3509	$\cdot 0360_6$	$1\cdot 4060_5$
(*f*)	·3114	$\cdot 0480_8$	1·4148
(*g*)	·3056	$\cdot 0504_8$	1·4159

From (*a*) and (*e*) $\lambda_1 = \cdot 1079\ \mu$ and $K = 5\cdot 119_7 \times 10^{-3}$.
,, (*b*) ,, (*e*)$\lambda_1 = \cdot 1077_9\ \mu$,, $K = 5\cdot 120_3 \times 10^{-3}$.
,, (*c*) ,, (*e*)$\lambda_1 = \cdot 1073_5\ \mu$,, $K = 5\cdot 125 \times 10^{-3}$.
,, (*d*) ,, (*e*)$\lambda_1 = \cdot 1075_3\ \mu$,, $K = 5\cdot 124_6 \times 10^{-3}$.
,, (*e*) ,, (*f*)$\lambda_1 = \cdot 1077_3\ \mu$,, $K = 5\cdot 120_4 \times 10^{-3}$.
,, (*e*) ,, (*g*)$\lambda_1 = \cdot 1076_7\ \mu$,, $K = 5\cdot 122 \times 10^{-3}$.

The mean value of $\lambda_1 = \cdot 1077\ \mu$, and of $K = 5\cdot 12_2 \times 10^{-3}$. The magneto-optical dispersion of normal propyl acetate is therefore represented by the equation

$$n\delta = 5\cdot 12_2 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1077)^2\}^2}.$$

The above equation was used to calculate the values of δ for other wave-lengths, and the results of the comparison are shown in Table III. (*c*).

The value of Verdet's constant for sodium light was also calculated from the equation representing the magneto-optical dispersion of normal propyl acetate and found

TABLE III. (*c*).

λ.	δ (observed).	δ (calculated).
·4621	$\cdot 0192_3$	$\cdot 0192_5$
·4430	$\cdot 0210_7$	$\cdot 0211_4$
·4142	$\cdot 0245_8$	$\cdot 0245_7$
·3980	$\cdot 0269_1$	$\cdot 0269_0$
·3860	$\cdot 0288_4$	$\cdot 0288_6$
·3727	$\cdot 0312_5$	$\cdot 0312_9$
·3609	$\cdot 0336_5$	$\cdot 0337_4$
·3450	$\cdot 0374_9$	$\cdot 0375_3$
·3364	$\cdot 0398_5$	$\cdot 0398_4$
·3250	$\cdot 0432_7$	$\cdot 0433_5$
·3146	$\cdot 0469_4$	$\cdot 0469_6$

to be ·01138 at 13° C. According to Perkin * the specific rotation of normal propyl acetate for sodium light at 13·5° C. is ·8670 †, and at 18·25° C. is ·8584. Taking these values it is estimated that the specific rotation of the liquid is ·8679 at 13° C. If Verdet's constant of water for sodium light be taken as ·01310, it is calculated that the value of Verdet's constant of normal propyl acetate for the same wave-length as deduced from Perkin's observations is given by ·1137 at 13° C. This result is in good agreement with that calculated from the magneto-optical dispersion equation representing the present observations.

* Perkin, *loc. cit.* p. 495.

† This is the mean of six values obtained at temperatures varying from 13·0° C. to 13·8° C.

Ordinary Dispersion.

The experimental results have shown that the magneto-optical dispersion of each liquid over the range of spectrum investigated can be represented by an equation of the type :

$$n\delta = K \frac{\lambda^2}{(\lambda^2-\lambda_1^2)^2},$$

which involves the presence of one absorption band of wave-length λ_1 situated in the Schumann-Lyman region of the spectrum.

It will be seen later that the ordinary dispersion of each liquid can be represented within experimental error by the equation :—

$$n^2-1=b_0+\frac{b_1}{\lambda^2-\lambda_1^2},$$

where λ_1 has the same value as deduced from the magneto-optical experiments.

If this assumption be made, the values of the two constants b_0 and b_1 can be determined from two values of the refractive index corresponding to two known wave-lengths.

Acetic Acid.

The values of the constants b_1 and b_0 calculated from pairs of values of n and λ are given in Table IV. (a). The value of λ_1 is taken as ·1042 μ.

Table IV. (a).

λ (microns).	n.	$b_1 \times 10^3$.	b_0.
·6563	$1{\cdot}3700_9$	$8{\cdot}973_4$	$\cdot 8557_8$
·4341	$1{\cdot}3806_9$		
·4341	$1{\cdot}3806_9$	$8{\cdot}986_5$	$\cdot 8557_1$
·2766	$1{\cdot}4115_9$		
·6563	$1{\cdot}3700_9$	$8{\cdot}983_2$	$\cdot 8557_6$
·2766	$1{\cdot}4115_9$		

The mean values of b_1 and b_0 are $8{\cdot}981\times10^{-3}$ and $\cdot8557_5$ respectively, and the dispersion of acetic acid at 19·5° C. is represented by the equation

$$n^2=1{\cdot}8557_5+\frac{8{\cdot}981\times10^{-3}}{\lambda^2-(\cdot1042)^2}.$$

This equation was used to calculate the values of n for certain wave-lengths in the region of the spectrum investigated, and the observed and calculated results are given in Table IV. (*b*).

TABLE IV. (*b*).

λ.	n (observed).	n (calculated).
·6678	1·3698	$1{\cdot}3698_1$
·5893	$1{\cdot}3720_3$	$1{\cdot}3720_2$
·5016	$1{\cdot}3758_7$	$1{\cdot}3758_8$
·4472	$1{\cdot}3795_8$	$1{\cdot}3795_8$
·4026	$1{\cdot}3838_6$	$1{\cdot}3838_5$
·3654	$1{\cdot}3888_3$	$1{\cdot}3888_7$
·3466	$1{\cdot}3920_6$	$1{\cdot}3921_0$
·3274	$1{\cdot}3960_3$	$1{\cdot}3960_5$
·2961	$1{\cdot}4044_3$	$1{\cdot}4045_1$
·2883	$1{\cdot}4071_5$	$1{\cdot}4071_4$
·2824	1·4093	$1{\cdot}4093_0$

Normal Propyl Acetate.

The values of constants b_1 and b_0 calculated from two pairs of values of n and λ are given in Table V. (*a*). The value of λ_1 is taken as $\cdot1077\ \mu$.

TABLE V. (*a*).

λ (microns).	n.	$b_1\times10^3$.	b_0.
·6678	1·3850	8·870	$\cdot8978_0$
·4375	1·3954		
·4713	1·3928	8·840	$\cdot8979_0$
·2961	$1{\cdot}4192_5$		

The mean values of b_1 and b_0 are $8{\cdot}855\times10^{-3}$ and

$\cdot 8978_5$ respectively, and the dispersion of normal propyl acetate at 13° C. is represented by the equation

$$n^2 = 1\cdot 8978_5 + \frac{8\cdot 855 \times 10^{-3}}{\lambda^2 - (\cdot 1077)^2}.$$

This equation was employed to calculate the values of n for certain wave-lengths in the region of the spectrum investigated, and the observed and calculated results are given in Table V. (*b*).

TABLE V. (*b*).

λ.	n (observed).	n (calculated).
·5876	1·3873	$1\cdot 3872_1$
·5016	1·3909	1·3909
·4922	1·3915	$1\cdot 3914_8$
·4472	1·3946	$1\cdot 3945_9$
·4023	1·3988	$1\cdot 3988_5$
·3700	1·4030	$1\cdot 4030_4$
·3530	1·4057	$1\cdot 4057_8$
·3274	1·4108	$1\cdot 4108_5$
·3065	1·4161	1·4162
·3036	1·4169	$1\cdot 4169_5$

Calculation of $\frac{e}{m}$.

The value of $\frac{e}{m}$ is given by the relation

$$\frac{e}{m} = -\frac{2KC^2}{b_1},$$

and can therefore be deduced from the results of the magneto-optical and ordinary dispersion experiments. In the above equation e is expressed in electrostatic units and the magnetic field in electromagnetic units. The values of K for acetic acid and normal propyl acetate, when δ is expressed in minutes per cm. gauss and λ in microns, are $4\cdot 715 \times 10^{-3}$ and $5\cdot 12_2 \times 10^{-3}$ respectively. When, however, δ is expressed in radians per cm. gauss and λ in cm., the corresponding values of K for the two liquids are $1\cdot 37_2 \times 10^{-14}$ and $1\cdot 49_0 \times 10^{-14}$. Similarly, the

corrected values of b_1 for acetic acid and normal propyl acetate, when λ is expressed in cm., are $\cdot 898_1 \times 10^{-10}$ and $\cdot 885_5 \times 10^{-10}$. The values of $\frac{e}{m}$ for acetic acid and normal propyl acetate, calculated from the above equation, are $\cdot 91_6 \times 10^7$ and $1 \cdot 01 \times 10^7$ e.m.u. respectively.

The discussion of the experimental results is postponed until further work on acetates, now in progress, has been completed.

Summary.

(1) The magneto-optical and ordinary dispersion of acetic acid are given by the equations :—

$$n\delta = 4 \cdot 71_5 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1042)^2\}^2},$$

and

$$n^2 - 1 = \cdot 8557_5 - \frac{8 \cdot 98_1 \times 10^{-3}}{\{\lambda^2 - (\cdot 1042)^2\}},$$

where n and δ are the values of the refractive index and Verdet's constant, and $\cdot 1042\,\mu$ the wave-length of the absorption band in the Schumann-Lyman region of the spectrum.

(2) The magneto-optical and ordinary dispersion of normal propyl acetate are given by the equations :—

$$n\delta = 5 \cdot 12_2 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1077)^2\}^2},$$

and

$$n^2 - 1 = \cdot 8978_5 + \frac{8 \cdot 855 \times 10^{-3}}{\{\lambda^2 - (\cdot 1077)^2\}},$$

where n and δ have the same significance as above, and $\cdot 1077\,\mu$ is the wave-length of the absorption band in the Schumann-Lyman region of the spectrum.

(3) The values of $\frac{e}{m}$ for acetic acid and normal propyl acetate calculated from the magneto-optical and ordinary dispersions are $\cdot 91_6 \times 10^7$ and $1 \cdot 01 \times 10^7$ e.m.u. respectively.

The authors wish to thank Prof. E. J. Evans, D.Sc., for his valuable help during the course of the investigation.

LXXXVI. *The Absolute Value of the Terms of* As.

By W. M. Hicks, *F.R.S.* *

In a recent communication to the Physical Society †, A. S. Rao has extended the analysis of the spectrum of neutral arsenic by the addition of a large number of multiplets. He has given an estimate of the absolute magnitude of the terms by assigning to $s^4 1$ a value of 85000, with a suggested possible error of ± 2000. It is the object of this note to obtain a closer value of these terms by the application of the oun-multiple law to the separations. The usual method of determining the oun is first to determine the true value of one term—say the lowest p—from the actual series observed. It is then possible to determine the denominator-differences—Δ—of the terms on which the separations depend. When the various Δ so determined have a considerable range it is easy to find the definite value of the oun itself. But the method is only applicable when the term values are known. Conversely, if the oun is already known, the various separations accurately observed, and the scheme of terms has been settled—that is, the true values are the scheme values + a common constant ξ,—it may be possible to argue back and determine the constant ξ. In the present case, however, no antecedent knowledge of the value of the oun is at disposal. It is then necessary to have recourse to the fact that the oun for any element always satisfies the law $\delta = 4$ ouns $= 361{\cdot}78\ (\mathrm{W}/100)^2$ with extreme exactness. For As, $\mathrm{W} = 74{\cdot}934$ (Aston), $74{\cdot}937$ (Krepelka) ‡, say $74{\cdot}935 \pm {\cdot}002$. This gives

$$\delta = 203{\cdot}149 \pm {\cdot}0027.$$

The coefficient of W^2 is correct within 1 in 18000 or δ within $\cdot 011$. We take $\delta = 203{\cdot}149 \pm {\cdot}014$. The δ lies between $203{\cdot}163$ and $203{\cdot}135$. All the following observational data are taken from Rao's paper.

To apply the method we must choose, in the first instance, terms of high value (small denominators) and accurate measures of the separations. In As we obtain them with combinations involving the terms designated by Rao as $p\mathrm{P}^2$, $s\mathrm{P}^1$, $p\mathrm{D}^2$, $s\mathrm{P}^2$. The data are—term values in italics, separations in thick type.

* Communicated by the Author.

† Proc. xliv. p. 594 (1932).

‡ Nat. cxxiii. p. 944 (1929).

		pD_2^2.		D_3^2.	pP_1^2.		P_2^2.
		74409·7	322	*74087·4*	*66816·7*	461	*66355·5*
$5sP_1^4$	*34308·0*	5, 40102·00		---	2, 32507·558	·181	4, 32046·377
		2492·89			3075·317		3119·576
	916	5·91			·457		·399
P_2^4	*33392·2*	4, 41017·91	·40	5, 40695·51	2, 33424·015	·239	4, 32962·776
		2437·22		2456·53	2990·991		3032·845
	1287	·51					·292
P_3^4	*32104·5*	3, 42305·42	·42	6, 41983·00	---		0, 34250·068
		2363·04		2381·19			2918·850
$5sP_1^2$	*31866·8*	9, 42543·23		---	7, 34949·272	·164	5, 34488·118
		2349·83			2860·452		2898·702
	1469	7·04			·962		·969
P_2^2	*30397·2*	3, 44013·27	·78	10, 43690·49	6, 36419·241	·172	8, 35958·067
		2271·39		2288·12	2744·991		2780·197

We start in all cases with the term values as given by Rao. As illustrating the method we take in more detail the case of $pP_2^2 = 66355{\cdot}5 + \xi$. The mean ν-separation of the four very accurate measures is 461·189, say 461·19. The two terms are then:

$$66355{\cdot}5 + \xi \quad (461{\cdot}19) \qquad 66816{\cdot}69 + \xi + d\nu$$

with denominators

$$1{\cdot}285649 - 9{\cdot}6880\,\xi \qquad 1{\cdot}281204 - 9{\cdot}5875(\xi - d\nu),$$

whence

$$\Delta^* = 4445 - {\cdot}1005\,\xi + 9{\cdot}587\,d\nu$$
$$= 21\tfrac{3}{4}\delta + 26 - {\cdot}1005\,\xi + 9{\cdot}587\,d\nu.$$

As a first approximation take $\xi = 260 + \xi$, the $d\nu$ is so small that at present it can be neglected. Then

$$\Delta = 21\tfrac{3}{4}\,\delta - {\cdot}1005\,\xi.$$

This must be a multiple of δ_1, say $m\delta_1$. Then

$${\cdot}1005\,\xi = m \times 50{\cdot}786$$
$$\xi = 502{\cdot}33\,m.$$

There remains, then only to test for a few values of m. A comparison with the results for the other separations—which it is not necessary to reproduce—shows that m must be 5 or 6, *i. e.*, ξ of the order 3000. This is so large that a recalculation is required, and it is found that a quantity

* It is to be remembered that in dealing with Δ, δ, &c., the mantissæ are to be regarded as multiplied by 10^6.

$2960+\xi$, where ξ is not large, must be added to Rao's values of the terms. The data are here given on this basis:

A. pP^2, 69315·5. $\nu=461·189$.
B. pD^2, 77047·4. $\nu=322·89$.
C. sP^2, 33557·2 $\nu=1469·96$
D. sP^4, 35064·5 $\nu_1=1287·29$; $\nu_2=916·43$; $\nu=2203·72$

322·39 is the mean of 33 values, $d\nu$ probably $<$ ·05.

1469·959 mean of two very accurate measures $=1469·96$.

2203·72 mean of two very accurate measures.

The calculations now give

A. $\Delta=4163·98=20\frac{1}{2}(203·121-·0045\,\xi+·45\,d\nu)$

B. $\Delta=2488·37=12\frac{1}{4}(203·132-·0040\,\xi+·63\,d\nu)$

C. $\Delta=38680=190\frac{1}{2}(203·045-·0089\,\xi+·142\,d\nu)$

D. $\begin{cases}\Delta_1=31597=155\frac{1}{2}\times 203·19\\ \Delta_2=21489=105\frac{3}{4}\times 203·20\\ \Delta\ =53086=261\frac{1}{4}(203·203-·0084\,\xi+·088\,d\nu)=261\frac{1}{2}(203·006\ldots)\end{cases}$

The numerical factors for Δ_1, Δ_2 are possibly wrong, for as they stand there is no triplet effect. Still the equality of δ in Δ_1, Δ_2 is striking, since a change of a unit in the multiple of Δ_1, Δ_2 alters δ by ·3 and ·5.

The next most significant sets are those of the lines in the ultra-red, since the O.E. of the ν are small.

		sP_1^4.		P_2^4.		P_3^4.
		34308·0	**916**	*33392·2*	**1287**	*32104·5*
pP_1^4	*24141·7*	80, 10166·27 9833·76		– – –		– – –
P_2^4	*23418·2*	– – –		100, 9973·35 10023·98		– – –
P_3^4	*22029·0*	– – –		– – –		150, 10074·81 9923·03
pD_1^4	*22975·5*	150, 11332·50 8821·76 **371·63**	**·45**	10, 10416·05 9597·94 **371·63**		– – –
D_2	*22603·9*	50, 11704·13 8541·65	**·45**	25, 10787·68 9267·29 **884·94**		– – –
D_3	*21718·9*	– – –		100, 11672·62 8564·71	**·68**	8, 10384·94 9826·69 **1167·02**
D_4	*20551·9*	– – –		– – –		100, 11551·96 8654·16

pD_2^2	*22447·4*	100, 11860·63 8428·94	**·50**	15, 10944·13 9134·81		– – –
				1614·82		
D_3^2	*20832·6*	– – –		25, 12558·95 7960·26	**·69**	100, 11271·26 8869·69

		$4sP_1^2$.	$4sP_2^2$.
pP_1^2	*20749·7*	50, 11188·15 8935·58	
		71·53	
P_2^2	*20678·0*	10, 11965·60 8355·00	

With $\xi = 2960 + \xi$ the terms and separations are:

E. pP_3^4 24989·0 $\nu_1 = 1389{\cdot}15$; $\nu_2 = 723{\cdot}08$; $\nu = 2112{\cdot}28$

F. pD_1^4 23511·9 $\nu_1 = 1167{\cdot}02$; $\nu_2 = 884{\cdot}94$; $\nu_3 = 371{\cdot}63$; $\nu = 2423{\cdot}59$

G. $5pD_3^2$ 23792·6 $\nu = 1614{\cdot}82$

H. pP^2 23638·0 $\nu = 71{\cdot}53$

In P^4 the ν are determined by using the ν of sP^4, *i. e.*, $2203{\cdot}69 + x$. Its error is then its own O.E.$+x$. The calculations now give

E. $\Delta_1 = 55911 = 275\frac{1}{4} \times 203{\cdot}13$

$\Delta_2 = 27386 = 134\frac{3}{4} \times 203{\cdot}23$

$\Delta = 83297 = 410(203{\cdot}163 - {\cdot}0117\,\xi + {\cdot}0905\,d\nu) = 410\frac{1}{4}(203{\cdot}040 + \ldots)$
$= 410\frac{1}{2}(202{\cdot}917 + \ldots)$

F. $\Delta_1 = 51686 = 254\frac{1}{2} \times 203{\cdot}09$

$\Delta_2 = 36809 = 181\frac{1}{4} \times 203{\cdot}08$

$\Delta_3 = 14894 = 73\frac{1}{4} \times 203{\cdot}33$

$\Delta = 103389 = 509(203{\cdot}122 - {\cdot}0123\,\xi + {\cdot}0779\,d\nu) = 509\frac{1}{4}(203{\cdot}022 + \ldots)$
$= 509\frac{1}{2}(202{\cdot}922 \ldots)$

G. $\Delta = 69350 = 341\frac{1}{2}(203{\cdot}22 - {\cdot}0153\,\xi + {\cdot}120\,d\nu) = 341\frac{3}{4}(202{\cdot}926 - \ldots)$

H. $\Delta = 3251{\cdot}7 = 16(203{\cdot}23 - {\cdot}0128\,\xi + 2{\cdot}835\,d\nu)$

For a satisfactory determination of the oun it is necessary to have data with a considerable range of multiples from low to high. The low multiples will then be quite definite, whilst a change in the high by a few units will only make a small change in the corresponding value of δ. This renders it possible to obtain greater accuracy when the possible $d\nu$ errors in the separations are small. Here are multiples ranging from $3\frac{1}{2}$ to 600. A change of unity in I. (see below) of $3\frac{1}{2}\,\delta$, *i. e.*, $14\,\delta_1$ to $15\,\delta_1$ alters the δ from 204·8 (within O.E.) to 215·5 (quite outside O.E.); or say from 49 to 50 in B alters δ from 203·132 to 199·0, again quite impossible.

A glance at the values of δ given above from A to H shows that while there is a very approximate agreement they cannot be all brought to a common value by allowing any possible values to the $d\nu$. It is then necessary to get a closer approximation to the value of ξ. This is easily attained by considering successive increments of (say) 5. The cases A, B, C show that ξ must be negative. The result shows that ξ must be close to -15. In the table below the values of δ thus modified are given in the second column of numbers. Two arrangements are also shown. The first to get a common value $\delta=203{\cdot}180$ and the second $203{\cdot}140$ with the $d\nu$ required. For $203{\cdot}140$ the corresponding $d\lambda$ are also given.

$\xi=2045$.

	Mult.		203·	$d\nu$.		203·	$d\nu$.	$d\lambda$.
A.	$20\frac{1}{2}$	203·188	·180	−·02		·143	−·1	·008
B.	$12\frac{1}{4}$	·192	·180	−·02		·142	−·08	·005
C.	$190\frac{1}{2}$	·178	·181	·02		·143	−·025	·002
D.	$(261\frac{1}{2})$	·132	·178	·5		·141	·1	−·008
E.	$(410\frac{1}{4})$	·216	·180	−·4	$(410\frac{1}{2})$	·138	·5	−·5
F.	$(509\frac{1}{4})$	·206	·180	−·33	$(509\frac{1}{2})$	·144	·5	−·4
G.	$341\frac{3}{4}$	·155	·179	+·2		·144	−·1	·06
H.	16	·2	·2	−·07		·14	−·09	·07

In estimating the $d\nu$, $d\lambda$ it must be remembered that they are distributed between two observations. The first four must be very accurate, and the $d\nu={\cdot}5$ in D will exclude the case of $\delta=203{\cdot}180$. It must also be remembered that the calculations are based on Rao's scheme of terms, which may quite possibly vary in different terms by a unit or more, *i. e.*, the ξ may vary by say ± 1 in different sets. The effect would be extremely small and only effective in the last steps of determining the limit with accuracy. In the high multiple cases, $\xi=1$ produces about the same effect as $d\nu={\cdot}1$. In D the accuracy depends on the measure of the faint line 0, 2918·850 on which an error of ·008 is quite conceivable. F depends on d^4 terms. It is shown in my 'Analysis of Spectra,' p. 192, that the R of d^2 terms in the spectra of neutral elements (I., III.) take a value of $R=R(1-{\cdot}000075)^2$, where R is the value for p terms. This effect is much too small to produce any appreciable effect on the value of δ unless Δ is very large, as it is here in F with $\Delta=103089$. The above value would make Δ less by 7 and δ by ·014. There is, however, no evidence as to the value of ζ in $R(1-\zeta)^2$ for d^4 terms.

The final conclusion, then, is that

$$\xi = 2945 \pm 3, \quad \delta = 203{\cdot}14 \pm {\cdot}01.$$

This value of ξ gives $s^4 1 = 85000 + 2945 = 87945$. This is equivalent to 10·85 volts, and closer to the value 11·52 ± ·5 referred to by Rao than his 85000 with 10·5 volts.

The remaining sets given by Rao have large O.E. $6sP^4$ are given as a succeeding order to $5sP^4$. Its separations are closely the same, and thus it cannot be successive ($\nu = 2304$, $\Delta = 137229 = 675\frac{1}{2} \times 203{\cdot}152$). The set, like many examples in other spectra, are probably not bi-term lines. His next set, marked B, has practically the same ν, and has the appearance of being a linked set. The $4dF^4$ are represented by three terms only and are consequently, owing to the multiplet effect, useless for our present purpose; dF^2 seems to be a single isolated doublet with no evidence for its allocation. Although the O.E. in these sets are large, it may be interesting to give the corresponding final ($\xi = -15$) values for them.

				dv
I.	$5sD^2$	27112·8	$\nu = 19{\cdot}34 \pm 1{\cdot}4$	
			$\Delta = 717 = 3\frac{1}{2}(205{\cdot}0 - {\cdot}0166\,\xi + 10{\cdot}5\,d_\nu)$	$-{\cdot}17$
J.	$5p^4P^4$	23967·5	$\nu_1 + \nu_2 = 1230{\cdot}64 \pm 1$	
			$\Delta = 52794 = 260\frac{1}{2}(203{\cdot}134 - {\cdot}0124\,\xi + {\cdot}16\,d\nu)$	·04
K.	pD^2	15342·7	$\nu = 89{\cdot}56 \pm 2$	
			$\Delta = 7748 = 38\frac{1}{4}(203{\cdot}20 - {\cdot}020\,\xi + 2{\cdot}27\,d\nu)$	$-{\cdot}02$
L.	dD^2	19546·3	mean $\nu = 100{\cdot}84$, best single 101·06	
			$\Delta = 6072 = 30(202{\cdot}86 - {\cdot}0155\,\xi + 2{\cdot}0\,d\nu)$	·14
M.	dD^4	21164·6	$\nu = 59{\cdot}86 + 240{\cdot}49 - 102{\cdot}38 = 197{\cdot}97 \pm 2$	
			$\Delta = 10549 = 52(203{\cdot}31 - {\cdot}015\,\xi + 1{\cdot}2\,d\nu)$	$-{\cdot}14$

In spite of their large possible O.E. the low multiples add weight to their evidence.

LXXXVII. *On the Eleven-year Period of Earthquake Frequency.* By Charles Davison, *Sc.D., F.G.S.**

THE attention of seismologists has naturally been directed to the possible existence of an eleven-year seismic period †. In a paper published a few years ago ‡, I investigated the period for several regions, the

* Communicated by the Author.

† See, for instance, Prof. L. A. Cotton in Amer. Seis. Soc. Bull. xii. pp. 56–58 (1922). The seismic maximum has been connected by some writers with the maximum, by others with the minimum, of sunspot frequency.

‡ Phil. Mag. vii. pp. 580–586 (1929).

materials being provided chiefly by Milne's great catalogue of destructive earthquakes *. The present paper contains a fuller examination of the subject based on this and other catalogues, the method used being that described in my earlier paper.

Of the following tables, the first contains the results for different degrees of intensity † in the two hemispheres and in continental areas; the second those for all three intensities of Milne's scale for smaller regions; and the third those for various countries depending on catalogues other than Milne's. The date of the epoch given in these three tables is that of the first in the eighteenth century, the interval included in each case being 1701–1898.

TABLE I.

Region.	Inten.	No. of shocks.	Epoch.	Ampl.
N. Hemisphere.	3	407	1709	·10
	2	623	1707	·19
	1	1318	1708	·10
	3, 2	1030	1708	·13
	3, 2, 1	2348	1708	·12
S. Hemisphere .	3	59	1711	·12
	2	71	1705	·31
	1	206	1709	·10
	3, 2	130	1705	·15
	3, 2, 1	336	ab. 1707	·06
Europe	3	76	..	..
	2	226	1709	·29
	1	727	1710	·12
	3, 2	302	1709	·22
	3, 2, 1	1029	1709	·13
Asia..........	3	211	1709	·25
	2	259	1708	·13
	1	422	1708	·18
	3, 2	470	1708½	·16
	3, 2, 1	892	1708	·16
N. America ..	3	90	1709	·18
	2	106	1703	·09
	1	167	1702½	·07
	3, 2	196	1711	·06
	3, 2, 1	363	ab. 1711	·06

* Brit. Ass. Rep., 1911, pp. 649–740.

† The scale of intensity used by Milne is the following:—

1. Walls cracked, chimneys broken, or old buildings shattered.
2. Buildings unroofed or shattered and some thrown down.
3. Towns destroyed and districts desolated.

Table II.

Region.	No. of shocks.	Epoch.	Ampl.
Central Europe	199	1708½	·09
Spain and Portugal	67	1710	·64
Italy	440	1709	·19
Balkan Peninsula	116	ab. 1710½	·15
Asia Minor	81	1710	·44
India............................	59	1709½	·23
Central Asia	72	1708	·23
China	163	1710	·25
Japan	86	1702	·19
Formosa	35	1709½	·38
Philippines......................	147	1708	·48
East Indies	133	1709	·32
Mexico	107	1702	·19
Central America	78	1708	·39
West Indies	79	1709	·20
S. America (south of equator)	159	1706	·28
Chile and Peru	115	1706	·19

The earthquakes considered in Table III., unlike those in all the other tables, did not, as a rule, reach destructive intensity. For Zante (Barbiani) two series of elements are given. The first are based on all the shocks recorded by him, the second on the more prominent shocks of which the separate times are entered. Of the two catalogues for Japan, that of S. Sekiya embraces all known earthquakes, that of F. Omori destructive earthquakes only *.

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9. Baratta, M., ' I Terremoti d'Italia,' pp. 3–621 (1901).
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12. Issel, A., and Agamennone, G., *Roma Uff. Centr. Met. Geod. Ann.* xv. pp. 38–71 (1893).

TABLE III.

Region.	Interval.	No. of shocks.	Epoch.	Ampl.
1. Norway (Kolderup)	1893–1925	503	1895 (1708)	·28
2. Finland (Henqvist)	1701–1898	128	1705	·63
3. Gt. Britain (Davison)	1701–1898	853	1709	·32
4. Switzerland (Swiss Seis. Com.)..	1883–1926	1445	1885 (1709)	·32
5. Neuberg (Gümbel)	1701–1887	151	1706	·13
6. Carinthia (Höfer)	1801–1887	134	ab. 1807 (1708)	·36
7. Andalusia (Mercalli)	1701–1876	91	1707	·20
8. Riviera (Mercalli)	1701–1876	579	1710	·75
9. Italy (Baratta)	1701–1898	690	1708	·13
10. Italy (Mercalli)	1501–1698	247	1501 (1710)	·09
11. Calabria (Mercalli)	1800–1887	660	1808 (1709)	·83
12. Zante (Agamennone)	1701–1854	92	1709	·25
13. Zante (Barbiani)	1826–1858	1541	1831 (1710)	·73
	1826–1858	1243	1831 (1710)	·56
14. Persia (Wilson)	1701–1898	71	1709	·53
15. India (Oldham)	1811–1865	280	1820 (1710)	·39
16. China (Drake)	1701–1898	75	1708	·20
17. Japan (Sekiya)	1701–1832	185	1707	·22
18. Japan (Omori)	1501–1698	69	1510 (1708)	·41
Japan (Omori)	1701–1898	62	1709	·20
19. Philippines (Masó)	1701–1898	155	1708	·41
20. New England (Brigham).......	1701–1865	210	1706	·35
21. California, etc. (Holden)	1811–1887	957	1819 (1709)	·19
22. New Zealand (Hogben)	1857–1889	642	1875 (1710)	·10

Footnote continued from p. 1087 :—

13. Barbiani, D. G., and B. A. Dijon, Acad. Mém. ii. pp. 165–193 (1852–53).
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20. Brigham, W. T., Boston Soc. Nat. Hist. Mem. ii. pp. 1–28 (1871).
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Duration of the Period.

The elements of the period for six successive centuries are given in Table IV., the earthquakes being those of destructive intensity in the Northern Hemisphere (Milne). The more accurate results for the early centuries are probably those for intensities 3 and 2 jointly. The first maximum after 1305 occurred in 1315, the last before 1898 in 1895. In this interval of 580 years, there were 53 periods, giving an average duration of 10·94 years. For earthquakes of all three intensities, the result is nearly the same, the first maximum occurring in 1314

TABLE IV.

Interval.	Inten.	No. of shocks.	Epoch.	Ampl.
1305–1403..	3, 2	82	1315	·26
	3, 2, 1	133	1314	·20
1404–1502..	3, 2	93	1413	·26
	3, 2, 1	176	1413	·15
1503–1601..	3, 2	130	1511	·22
	3, 2, 1	222	ab. 1507	..
1602–1700..	3, 2	225	ab. 1612	·16
	3, 2, 1	342	ab. 1612	·09
1701–1799..	3, 2	294	1709	·26
	3, 2, 1	584	1708½	·12
1800–1898..	3, 2	743	1807	·12
	3, 2, 1	1798	1807	·12

and the last in 1895. The mean duration of the 53 periods is 10·96 years.

The mean length of the period may also be estimated by taking five-yearly means of the numbers of earthquakes of all three intensities in successive years*. The result shows a series of maxima, the intervals between which occasionally vary in length, although, as a rule, they are nearly equal. The first maximum after 1301 occurred in 1303, the last before 1890 in 1883. Thus, in 580 years, there were 53 periods, giving an average duration of 10·94 years.

* This method is, of course, the same as that used elsewhere in this paper, but, being applied to single successive intervals of eleven years, it is affected by the grouping of earthquakes connected with other periods, such as that of nineteen years.

Epochs of the Period.

Throughout the Northern Hemisphere the epochs of the period cluster in or about the years 1708 and 1709. The average of all is about 1708½.

It is doubtful if, and how far, the exceptional epochs have any significance. In North America the epoch for intensity 3 is normal (1709). For Japan, Milne's catalogue of destructive earthquakes gives 1702 as the epoch; Omori's, which is somewhat less full, 1709 for the same interval, and 1510 (or 1708) for the preceding two centuries. In Mexico the epoch for earthquakes of intensity 1 is 1711, though the epoch for intensities 3 and 2 is 1703.

For the Southern Hemisphere, taking destructive earthquakes of all intensities, we have the following dates for the maximum epoch: about 1707 for the whole hemisphere, 1706 for South America (south of the Equator), 1706 for Chile and Peru (1710 for Chile alone), and 1709 for the East Indies; also 1875 (1710) for New Zealand (Hogben). As the average of the epochs for South America, the East Indies, and New Zealand is about 1708½, it would seem that the epoch of the eleven-year period is roughly constant all over the earth. And this inference is strongly supported by the dates given in Tables V. and VI.

Connexion between the Eleven-year Periods of Earthquake and Sunspot Frequency.

The following Table V. contains the elements of the sunspot period for the years 1867–1899 and of the seismic

TABLE V.

	Intensity.	No. of shocks.	Epoch.	Ampl.
Sunspots	..	..	1860 or 1860½	·97
Earthquakes.				
N. Hemisphere..	3, 2	446	1861	·18
	3, 2, 1	1126	1861½	·12
S. Hemisphere..	3, 2	74	1860	·15
	3, 2, 1	232	1861	·09

period for 1856–1899, the years during which our earthquake records are most complete.

Again, taking five-yearly means of successive annual numbers of sunspots and of the earthquakes in both hemispheres, we obtain the years of greatest and least frequency given in Table VI. Most of the elements depend on earthquakes of all three intensities. In a few cases, however, the epochs are not well defined, and these are supplied by the series of earthquakes of intensities 3 and 2 jointly and of intensity 1, the former being denoted by an asterisk and the latter by a dagger. The dates given for the other epochs by all three series agree closely.

TABLE VI.

		Earthquakes.			
Sunspots.		N. Hemisphere.		S. Hemisphere.	
Max.	Min.	Max.	Min.	Max.	Min.
1829		1829		1829	
	1833		1834		
					. .
					1832½†
1837		. .		. .	
		1837†		1836½†	
	1843		1842		1840
1848		1846		1847	
				1849*	
	1855		. .		. .
			1854†		1854†
1860		1860		1861	
				1860*	
	1866		1865		1865
1871		. .		1870½	
		1870*		1871†	
	1877		1877		. .
					1877*
1883		1883		. .	
				1883*	
	1888		1891		1891
1894		1895		1896	
				1894½*	

It follows from these tables (i.) that the epochs of the eleven-year seismic period are the same in both hemispheres, and (ii.) that they agree closely with the corresponding periods of sunspot frequency.

LXXXVIII. *On the Nineteen-year Period of Earthquake Frequency and on its Connexion with the Nutation Period of the Earth.* By CHARLES DAVISON, *Sc.D., F.G.S.**

THE existence of a nineteen-year seismic period was noticed in a paper published a few years ago †. The present paper continues the subject in greater detail. It is convenient to retain the above name for the period, and, in working, to regard its duration as exactly 19 years, though, as will be seen, it is somewhat less, namely, 18·6 years.

Of the following tables, the third is based on various regional lists, the rest on Milne's great catalogue of destructive earthquakes. The object of Table I. is to show that, for continental areas, earthquakes of different intensities have approximately the same maximum epoch

TABLE I.

Region.	Inten.	No. of shocks.	Epoch.	Ampl.
N. Hemisphere	3	404	1717	·29
	2	597	1715	·23
	1	1235	1714	·23
	3, 2	1001	1715½	·23
	3, 2, 1	2236	1714	·21
Europe	3	67	1714	·35
	2	225	1715	·18
	1	667	1715	·17
	3, 2	292	1714	·21
	3, 2, 1	959	1714	·23
Asia	3	197	1717	·35
	2	240	1714	·32
	1	406	1714	·26
	3, 2	437	1716	·27
	3, 2, 1	843	1716	·23
N. America	3	88	1716½	·27
	2	103	1716	·26
	1	150	1718½	·23
	3, 2	191	1715	·26
	3, 2, 1	341	1716½	·20

* Communicated by the Author.

† Phil. Mag. vii. pp. 580–586. (1929).

for the period. In Tables II. and IV. earthquakes of intensities 3 and 2 (Milne scale) only are included. The date of the epoch given in Tables I.–III. is that of the first in the eighteenth century, the interval used in Tables I. and II. being 1701–1890.

TABLE II.

Region.	No. of shocks.	Epoch.	Ampl.
Spain and Portugal.......	27	1715	·85
China..................	88	1715½	·63
Japan	55	1705	·47
Philippines..............	68	1714	·32
East Indies..............	34	1715½	·65
Mexico	64	1715	·38
West Indies	44	1713	·59
Central America	55	1705	·29
S. America	66	ab. 1707	·42
Chile and Peru...........	43	1704½	·62
North Temperate Zone ...	360	1716	·24
North Tropics	241	1716	·18

TABLE III.

Region.	Interval.	No. of shocks.	Epoch.	Ampl.
Italy (Baratta)	1701–1890	597	1714	·09
Gt. Britain (Davison, ints. 9–4)......................	1701–1890	118	1716	·47
Persia (Wilson)	1701–1890	65	1714	·45
India (Oldham)	1717–1869	336	1715	·42
China (Drake)	1701–1890	72	1717	·69
Japan (Sekiya)	1701–1852	187	1716	·18
Japan (Omori)	1701–1890	51	1705	·56
Philippines (Masó)	1701–1909	181	1718	·15
Central America (Montessus) .	1701–1890	282	1718	·27

Duration of the Period.

It has been assumed so far that the duration of the period is exactly 19 years. In reality, it is somewhat

less. In the next table are given the elements of the nineteen-year period for six successive intervals of 95 years each.

TABLE IV.

Interval.	No. of shocks.	Epoch.	Ampl.
1321–1415	75	ab. 1330	·27
1416–1510	105	1427	·32
1511–1605	117	1526½	·26
1606–1700	219	1619	·21
1701–1795	291	1715	·24
1796–1890	678	1811	·23

Thus, for earthquakes of intensities 3 and 2, the first epoch after 1321 fell in 1330, and the last before 1890 in 1887. In this interval of 557 years, there were 30 complete periods. This gives an average duration for the period of 18·6 years.

Again, taking nine-yearly means of the numbers of shocks in successive years in the Northern Hemisphere (Milne, ints. 3–1), we find a succession of maxima, not always at quite equidistant intervals, the first clearly-marked after 1321 being in 1348 and the last before 1890 in 1884. In this interval of 536 years, there were 29 complete periods, giving for the period an average duration of 18·5 years. Thus, it would seem that the length of the period is 18·6, rather than 19 years.

Epochs of the Period.

All over the Northern Hemisphere, except perhaps in Japan and Central America, the maximum epochs cluster about the years 1714–1717, with an average of about 1715½. In Table II. the epochs for the above countries fall in 1705, while Omori's catalogue of destructive Japanese earthquakes (Table III.) also gives the same epoch. It should be noticed, however, that the lists of Sekiya and Montessus, which include non-destructive shocks, give 1716 as the epoch for Japan and 1718 as that for Central America. That the epoch does not vary

with the latitude in the Northern Hemisphere is also shown by the last two lines of Table II.

For the Southern Hemisphere, the materials are less abundant. From the above analysis, the epoch seems to fall during the interval 1704–1707, with an average of about 1705½, that is, the epochs in the two hemispheres are reversed. This is shown more clearly by taking nine-yearly means of the numbers of earthquakes of all three intensities during successive years. The dates of the maximum and minimum epochs, which are given in Table VI., cannot lay claim to great accuracy, but they are sufficient to show the correspondence between the maxima in one hemisphere and the minima in the other.

Origin of the Nineteen-year Period.

The seismic period here considered seems to be closely connected with the nutation period of the earth:

(i.) The two periods have the same duration, namely, 18·6 years.

(ii.) Their maximum epochs are nearly or quite coincident.

Table V. contains their dates for the interval 1824–1899, during which our record of destructive earthquakes is fairly complete, and for intensities 3–1.

TABLE V.

	No. of shocks.	Epoch.	Ampl.
Nutation.			
$\Delta\omega$ negative	..	1829½	..
$\Delta\omega$ positive	..	1839	..
Earthquakes.			
N. Hemisphere	1557	1830	·11
S. Hemisphere..........	297	1839	·15

Again, taking nine-yearly means of successive annual numbers of earthquakes in both hemispheres during the nineteenth century, we have the years of greatest and least frequency given in Table VI. The table on the

whole depends on earthquakes of all three intensities. Similar tables founded on earthquakes of intensities 3–2 and intensity 1 have also been prepared, and when the epochs differ from those for intensities 3–1, they are added below, the former being denoted by an asterisk and the latter by a dagger. In the first columns, the epochs for nutation are given to the nearest half-year.

Table VI.

Nutation.		Earthquakes.			
		N. Hemisphere.		S. Hemisphere.	
$\Delta\omega$ neg.	$\Delta\omega$ pos.	Max.	Min.	Max.	Min.
1811		1810			1808
		1811½*			1810*
	1820		1819½	1818	
				1820*	
1829½		1827½			1827½
		1828½*			
	1839		1836	1835	
			1839*	1837†	
1848		..			..
		1850*			1850†
	1857½		1858	1861	
				1860†	
1866½		1865			1868
					1866*
	1876		1875	1872	
			1876*	1875*	
1885½		1884			1883
		1885*			1883½*
	1894½		1894	1893	
			1894½*		

The stresses that result in nutation may seem small to affect the frequency of earthquakes so great as those of the highest degree of Milne's scale. But, as Prof. Milne showed in 1900, and as Prof. H. Nagaoka has recently illustrated *, the occurrence of great earthquakes is closely connected with the epochs of other minute changes in the direction of the earth's axis.

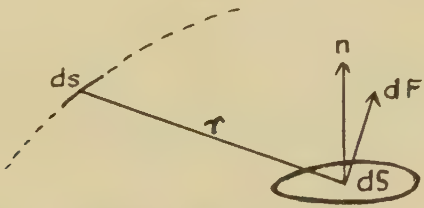
* J. Milne, Brit. Ass. Rep. 1900, pp. 107–108; 1903, pp. 78–80; 1906, pp. 97–99. F. Omori, Imp. Earthq. Inv. Com. Publ. no. 18, 1904, pp. 13–21. C. G. Knott, Brit. Ass. Rep. 1907, pp. 91–92. H. Nagaoka, Tokyo Imp. Acad. Proc. xiii. pp. 284–287 (1932), and 'Nature,' cxxx. p. 541 (1932).

LXXXIX. *On some new Formulæ for the Calculation of Self and Mutual Induction of Coaxial Circular Coils in Terms of Arithmetico-geometrical Scales.* By Louis V. King, *F.R.S., Macdonald Professor of Physics, McGill University, Montreal, Canada* *.

Section 1.—*Derivation of the Fundamental Formula.*

THE magnetic force $d\mathrm{F}$ due to current i flowing in an element ds of a circuit (fig. 1) is given by Ampère's expression

Fig. 1.



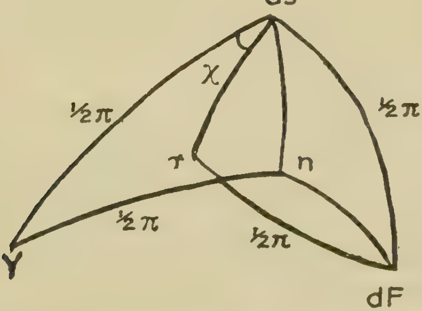
$$d\mathrm{F}=i\,ds\,.\sin\,(\widehat{r\,.\,ds})/r^2, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and is perpendicular to both r and ds. The component perpendicular to an element of area $d\mathrm{S}$ is $d\mathrm{F}\,.\cos\,(\widehat{n\,.\,d\mathrm{F}})$, where n is the normal to $d\mathrm{S}$. It thus follows that the contribution of the current element $i\,ds$ to the magnetic flux through $d\mathrm{S}$ is given by

$$d\Phi=i\,ds\,.\,d\mathrm{S}\cos\,(\widehat{n\,.\,d\mathrm{F}})\,.\sin\,(\widehat{r\,.\,ds})/r^2. \quad . \quad . \quad (2)$$

If from a given point we draw lines parallel to the directions of ds, $d\mathrm{F}$, r, and n to intersect a sphere, we obtain the spherical triangles of fig. 2.

Fig. 2.



* Communicated by the Author.

Let Y be a direction perpendicular to both ds and n. We then have from the spherical triangle $(n.ds.d\mathrm{F})$,

$$\cos(\widehat{n.d\mathrm{F}})=\cos\chi.\sin(\widehat{n.ds}),$$

and from the triangle (r, Y, ds),

$$\cos(\widehat{r.\mathrm{Y}})=\cos\chi.\sin(\widehat{r.ds}).$$

Equation (2) then becomes

$$d\Phi=i\,ds.\sin(\widehat{n.ds}).(d\mathrm{S}/r^2)\cos(\widehat{r.\mathrm{Y}}). \quad . \quad . \quad (3)$$

But the term

$$d\mathrm{Y}=(d\mathrm{S}/r^2).\cos(\widehat{r.\mathrm{Y}})$$

may be interpreted as the component perpendicular to n and ds of the gravitational attraction at ds of the element of area dS, supposed to be of unit surface density. We may thus write (3) in the form

$$d\Phi=i\,ds.\sin(\widehat{n.ds}).d\mathrm{Y}. \quad . \quad . \quad . \quad . \quad (4)$$

If one of the circuits encloses a plane area of which dS is an element, and if ds is a line-element of the second circuit, it follows from (4) that the coefficient of mutual induction of the two is given by

$$\mathrm{M}=\int \mathrm{Y}.\sin(\widehat{n.ds}).\,ds, \quad . \quad . \quad . \quad . \quad (5)$$

where Y is the component at ds perpendicular to dS and parallel to the plane of the second circuit, of gravitational attraction of matter of unit surface density uniformly distributed over the plane of the second circuit.

In nearly all cases occurring in practice, formula (5) is immediately applicable to the derivation of a formula for M, making use of well-known results from the theory of gravitational attractions.

Section 2.—*Mutual Inductance of two Coaxial Circles.*

Let the radius of the larger circle be A, that of the smaller a, and d the distance between their planes. Applying (5) we note that $\sin(\widehat{n.ds})=1$, and that Y is a constant over the circumference of the circle of which ds is an element. Thus,

$$\mathrm{M}=2\pi\mathrm{AY}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To calculate Y, divide the circle into rods as indicated in fig. 3, and make use of the well-known formula,

$$dY=(\cos\alpha_1-\cos\alpha_2)\frac{dm}{d},$$

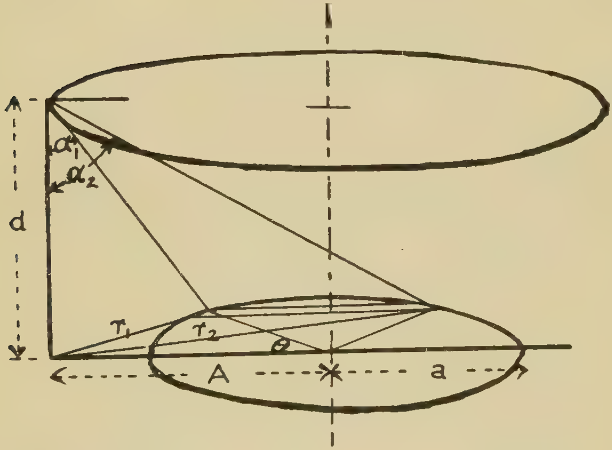
where dm is the mass per unit length of one of the rods, given by

$$dm=d(a\sin\theta)=a\cos\theta\,.\,d\theta.$$

Note that

$$\cos\alpha_1=\frac{d}{(r_1^2+d^2)^{\frac{1}{2}}},\qquad \cos\alpha_2=\frac{d}{(r_2^2+d^2)^{\frac{1}{2}}},\qquad (6.1)$$

Fig. 3.



where

$$r_1^2=a^2+A^2-2aA\cos\theta \text{ and } r_2^2=a^2+A^2+2aA\cos\theta.$$

We easily derive the formula,

$$Y=2a\int_0^{\pi}\frac{\cos\theta\,d\theta}{(d^2+a^2+A^2-2aA\cos\theta)^{\frac{1}{2}}}.\qquad (6.2)$$

If we now write $\cos\frac{1}{2}\theta=\mathrm{sn}\,(u,k)$, where

$$\left.\begin{aligned}k^2&=4Aa/\{(a+A)^2+d^2\},\\ k'^2&=\{(A-a)^2+d^2\}/\{(A+a)^2+d^2\},\end{aligned}\right\}\qquad (7)$$

or

$$k'=R_2/R_1,$$

R_1 and R_2 being the greatest and least distances respectively of a point on one circle to the circumference of the other, formulæ (6) and (6.2) give the result

$$M=4\pi\sqrt{Aa}\int_0^K (2\,\mathrm{sn}^2 u-1)\,du=\frac{8\pi\sqrt{Aa}}{k}K\left\{\frac{K-E}{K}-\tfrac{1}{2}k^2\right\}, \quad . \; . \; . \quad (8)$$

which is Maxwell's first formula *.

If the circle of fig. 3 be divided up into rods at right angles to the direction of Y, it may easily be shown that Maxwell's second formula is directly obtained,

$$M=8\pi(Aa/k_1)^{\frac{1}{2}}\{K(k_1)-E(k_1)\}, \quad . \; . \; . \quad (9)$$

in which $K(k_1)$ and $E(k_1)$ are the complete elliptic integrals to modulus k_1 given by

$$k_1=(R_1-R_2)/(R_1+R_2). \quad . \; . \; . \; . \; . \quad (10)$$

Formula (9) is connected with (8) by Landen's transformation.

For purposes of numerical computation, both formulæ are included in an extremely elegant expression arising from the formation of the scale of *arithmetico-geometrical means*. If we start with the positive numbers

$$(a_0=1, \quad b_0=k', \quad c_0=k),$$

and form successively

$$\left.\begin{array}{lll} a_0=1, & b_0=k', & c_0=k, \\ a_1=\frac{1}{2}(a_0+b_0), & b_1=\sqrt{a_0 b_0}, & c_1=\frac{1}{2}(a_0-b_0). \\ a_2=\frac{1}{2}(a_1+b_1), & b_2=\sqrt{a_1 b_1}, & c_2=\frac{1}{2}(a_1-b_1). \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_n=\frac{1}{2}(a_{n-1}+b_{n-1}), & b_n=\sqrt{a_{n-1}b^{n-1}}, & c_n=\frac{1}{2}(a_{n-1}-b_{n-1}), \\ \cdots & \cdots & \cdots \end{array}\right\} \quad . \; . \; . \quad (11)$$

the a's and the b's tend with extreme rapidity to the same limit denoted by

$$M(a_0, b_0)=\operatorname*{Lt}_{n\to\infty} a_n, \quad . \; . \; . \; . \; . \quad (12)$$

even when a_0 and b_0 are initially numbers of very different magnitudes.

* Maxwell, 'Electricity and Magnetism,' 3rd ed. p. 339 (1904).

It is easily seen that if we commence with

then
$$\left.\begin{matrix} A_0=\varepsilon a_0, & B_0=\varepsilon b_0, & C_0=\varepsilon c_0, \\ A_n=\varepsilon a_n, & B_n=\varepsilon b_n, & C_n=\varepsilon c_n, \end{matrix}\right\} \quad . \quad (13)$$

for all values of n.

The array of numbers (11) will be referred to as the *scale* of *arithmetico-geometrical means* (a_0, b_0), or, more briefly, as the A.G.M. scale (a_0, b_0). Also, the limit $M(a_0, b_0)$ will be denoted by a_n as long as by so doing no ambiguity is involved.

It follows from (13) that

$$M(\varepsilon a_0, \varepsilon b_0)=\varepsilon M(a_0, b_0). \quad . \quad . \quad . \quad . \quad (14)$$

It may be proved that if we start with the A.G.M. scale $(a_0=1, \ b_0=k')$, the complete elliptic integrals K and E are given by

$$\left.\begin{aligned} &K=\tfrac{1}{2}\pi/a_n \\ \text{and} \\ &(K-E)/K=\tfrac{1}{2}(c_0^2+2c_1^2+4c_2^2+\ldots+2^n c_n^2+\ldots), \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

while

$$a_n K' = \log\frac{4a_1}{c_0} - \sum_1^\infty (\tfrac{1}{2})^n \log\frac{a_n}{a_{n+1}}. \quad . \quad . \quad (16)$$

This, with Legendre's relation,

$$\frac{E}{K}+\frac{E'}{K}-1=\frac{\frac{1}{2}\pi}{KK'},$$

enables us to compute the complementary integrals K′ and E′.

It thus follows, writing $k^2=c_0^2$, that (8) may be written

$$M=8\pi\sqrt{Aa}/k^2\{\tfrac{1}{2}\pi/a_n\}[c_1^2+2c_2^2+\ldots+2^{n-1}c_n^2+\ldots]. \quad . \quad . \quad . \quad (17)$$

For the purposes of computation, it is more convenient to form the A.G.M. scale

$$(A_0=R_1, \ B_0=R_2).$$

Since

$$M(1, k')=M(1, R_2/R_1)=(1/R_1).M(R_1, R_2),$$

by (7) and (14), and

$$c_n=(1/R_1)C_n$$

by (13), (17) finally gives the formula

$$M=\frac{2\pi^2}{A_n}[C_1^2+2C_2^2+4C_3^2+\ldots+2^{n-1}C_n^2+\ldots], \quad (18)$$

a most elegant series first obtained by the writer in 1921 *.

It is readily seen that the same series is obtained from Maxwell's second formula (9), since k' and k of that formula are b_1, and c_1, of the A.G.M. scale (11). This is readily understood from the fact that the formulæ (15) are derived from the repeated application of Landen's transformation.

The formula (17) is extremely convergent and convenient to use, inasmuch as the A.G.M. scale on which it depends may be rapidly obtained by means of a modern calculating machine to almost any degree of accuracy, without the use of tables †. To illustrate the numerical work we take an example from the Bulletin 169 (third edition) of the U.S. Bureau of Standards.

* King, L. V., Proc. Roy. Soc. A, c. p. 63 (1921).

† The writer has found a calculating machine of the Odhner type (Brunsviga Calculator) very suitable for the square-rooting processes involved in the formation of the A.G.M. scales by the ordinary arithmetical rule. In carrying out the process, the work is considerably shortened by keeping in mind the rule that of the first p digits out of the number n required in the square root have been obtained by the usual process, the next $p-1$ digits can be obtained by division only, with a possible error of 1 in the last digit (Chrystal's 'Algebra,' Part 8, 5th edition, 1904, p. 210).

In many cases the combined use of L. M. Milne-Thomson's 'Standard Table of Square Roots' (London, Bell, 1929, 6×10 cloth, pp. x+100) results in reduced labour.

In working with trigonometrical recurrence formulæ, the use of tables adapted to the centesimal division of the quadrant are recommended, such as those of Roussilhe and Brandicourt, 'Tables à 8 décimales des valeurs naturelles des sinus, cosinus et tangentes dans le systeme décimal, de centigrade en centigrade.' Section de Géodésie de l'Union Géodésique et Géophysique Internationale. Publication Speciale no. 1. (Paris : Dorel, 1925, 7×10, cloth, pp. 139.)

For further information on mathematical tables and calculating machines, the computer should consult an article by L. J. Comrie, "Mathematical Tables" ('Monthly Notices of the Royal Astronomical Society,' February 1932, vol. xcii. no. 4), or address inquiries to H.M. Nautical Almanac Office, Royal Naval College, London, S.E. 10).

Let $A=a=25$ cm., $d=1$ cm., $R_1=50{\cdot}010000$, $R_2=1{\cdot}000000$.

(*n*).	A*n*.	B*n*.	C*n*.	C*n*².
(0)	50·010000	1·000000		
(1)	25·505000	7·071774	24·50500	600·4950
(2)	16·288387	13.430000	9·21661	84·9459
(3)	14·859193	14·79030	1·42919	2·0426
(4)	14·82475	14·82471	·03445	·0012
(5)	14·82473	14·82473	·00002	·0000

Formula (17) gives

$$M=\frac{2\pi^2\times 778{\cdot}5666}{14{\cdot}82473}=1036{\cdot}666 \text{ cm.},$$

agreeing with the value 1036·6663 obtained from Legendre's tables.

It will be evident from this somewhat extreme case that the single formula (17) is convenient to employ in all cases likely to arise in practice, and may thus be considered as a standard formula, replacing a large number of special formulæ designed to meet varying conditions.

When the circles are very close together, k' is small, and the A.G.M. scale (11) may conveniently be replaced by the complementary scale

$$(a_0'=1,\ b_0'=k,\ c_0'=k'),$$

which is then extremely convergent.

Making use of (16) together with Legendre's relation, we find that in terms of the complementary scale (denoted by accented letters), we have,

$$K-E=\frac{1}{2a_n'}\left\{\log\frac{4a_1'}{c_1'}-\log\frac{a_1'}{a_2'}-\tfrac{1}{2}\log\frac{a_2'}{a_3'}\ldots\right\}$$
$$\{1-\tfrac{1}{2}c_0'^2-c_1'^2-2c_2'^2-4c_3'^2-\ldots\}-a_n'. \quad \ldots \quad (19)$$

In applying this formula, it is most convenient to make use of (9) and the A.G.M. scale

$$(A_0'=R_1+R_2,\ B_0'=R_1-R_2),$$

which may be written out as follows :—

$$\left.\begin{array}{llll} (n). & A_n'. & B_n'. & C_n'. \\ 0\ldots\ldots & R_1+R_2 & R_1-R_2 & 2\sqrt{R_1R_2} \\ 1\ldots\ldots & R_1 & 2\sqrt{Aa} & R_2 \\ 2\ldots\ldots & \tfrac{1}{2}R_1+\sqrt{Aa} & (4AaR_1{}^2)^{\frac{1}{4}} & \tfrac{1}{2}R_1-\sqrt{Aa} \end{array}\right\} . \quad \ldots \quad (20)$$

Inserting a few terms from the A.G.M. scale (2), this formula may be conveniently written

$$M=2\pi A_n'\left[\frac{(R_1{}^2-2C_2'^2-4C_3'^2-\ldots)}{A_n'^2}\left\{\log\frac{4A_2'}{R_2}-\tfrac{1}{2}\log\frac{A_2'}{A_3'}-\tfrac{1}{4}\log\frac{A_3'}{A_4'}-\ldots\right\}-2\right]. \quad \ldots \quad (21)$$

Applying this formula to the example just considered, we have the complementary A.G.M. scale as follows :—

(n).	A_n'.	B_n'.	C_n'.
1	50·010000	50·000000	1·000000
2	50·005000	50·005000	·005000

Formula (21) gives for M,

$$M=2\pi\left[\frac{2501\cdot000}{50\cdot00500}\log_e 200\cdot0200-100\cdot0100\right]$$

$$=1036\cdot664_9 \text{ cm.}$$

Since $A_n'=A'_{n+1}+C'_{n+1}$, the logarithmic terms ultimately converge to the same degree as the sequence $C_2', C_3', C_4' \ldots C_n'$. Logarithms to the base 10 may be used in computing successive terms, and the entire series then multiplied by the factor

$$\mu=\log 10=2\cdot30258509$$

to convert the logarithms to base *e*.

For accuracy, the first term of (21) should be about twice the second. To find the approximate value of R_1/R_2 corresponding to this condition, we write

$$R_1{}^2\log(4A_1'/R_2')=4A_n'^2,$$

or since in these circumstances R_1 and A_n are not very different, we may write approximately,

$$\log(4A_1'/R_2') \sim 4, \qquad 4A_1'/R_2 \sim e^4 \sim 54 \cdot 6, \qquad A_1'/R_2 \sim 13 \cdot 6,$$

or, since

$$A_1' = \tfrac{1}{2}R_1 + \sqrt{Aa},$$

and

$$2\sqrt{Aa} = (R_1^2 - R_2^2)^{\frac{1}{2}},$$

we have

$$\{R_1 + (R_1^2 - R_2^2)^{\frac{1}{2}}\}/R_2 \sim 27 \cdot 3,$$

giving

$$R_1/R_2 \sim 13.$$

Hence, we may consider the formula (21) to be suitable for calculation for the range

$$0 < (R_2/R_1) < 0 \cdot 1.$$

It may without serious loss of accuracy be extended to the range

$$0 < (R_2/R_1) < 0 \cdot 2.$$

For seven-figure accuracy, the A.G.M. scale need then only be calculated to A_2'.

Section 3.—*Mutual Inductance of a Circle and a Coaxial Helix.*

Let A be the radius of the cylinder on which the helix is traced, a that of the coaxial circle. Measuring z from the plane of the circle, let $z = d_1$ and $z = d_2$ be the positions of the extremities of the helix.

Referring to the fundamental formula (5), we have

$$\sin \widehat{(n.ds)} = \sin \epsilon,$$

where ϵ is the angle which the helix makes with the generating lines of the cylinder. Since $dz = ds \,.\, \cos \varepsilon$, equation (5) gives

$$M = \tan \epsilon \int_{d_1}^{d_2} Y \, dz. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The term under the integral sign represents the component of attraction, by matter of unit surface density, distributed over the area of the circle in a direction parallel to its plane, on a rod of unit line density situated on one of the generating lines of the cylinder,

and having its ends at distances d_2 and d_1, from the plane of the circle. If $d\sigma$ be an element of area at P (fig. 4 *a*) in the plane of the circle, the attraction along OP by the rod A_1A_2 is, from a simple theorem in attractions,

$$(\sin\alpha_2 - \sin\alpha_1)\, d\sigma/r.$$

Fig. 4 *a*.

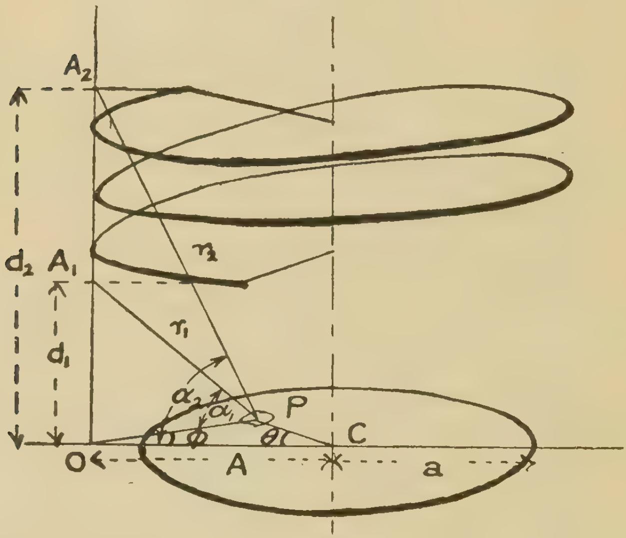
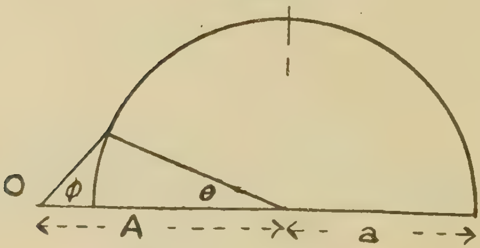


Fig. 4 *b*.



Resolving along OP which makes an angle ϕ with OC, and integrating over the area of the circle, we find that

$$M = \tan\epsilon\,(I_2 - I_1), \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where

$$I = \int \sin\alpha \cos\phi \, d\sigma/r, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the suffixes 1 and 2 refer to the values of the variables defining the position of the ends A_1 and A_2 of the rod.

Writing

$$d\sigma = r\,dr\,d\phi$$

in (24), and integrating with respect to ϕ, we have,

$$I = \int \sin\alpha \sin\phi\, dr, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where (r, ϕ) now refer to a point on the circumference of the circle of radius a (fig. 4 b).

Changing the variable to θ, where

$$r^2 = A^2 + a^2 - 2Aa\cos\theta,$$

we have

$$r\,dr = Aa\sin\theta\, d\theta, \qquad r\sin\phi = a\sin\theta,$$

and

$$\sin\alpha = d/(r^2+d^2)^{\frac{1}{2}}.$$

Integrating to include the entire circumference of the circle, we find that

$$I = 2Aa^2 d\int_0^{\pi} \frac{\sin^2\theta\, d\theta}{(a^2+A^2-2Aa\cos\theta)(d^2+a^2+A^2-2Aa\cos\theta)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad (26)$$

To reduce the integral, write $\cos\frac{1}{2}\theta = \operatorname{sn}(u, k)$, where

$$k^2 = \frac{4Aa}{(A+a)^2+d^2} \quad \text{and} \quad k'^2 = \frac{(A-a)^2+d^2}{(A+a)^2+d^2} = \frac{R_2^2}{R_1^2}, \quad . \quad . \quad . \quad (27)$$

R_1 and R_2 being the greatest and least distances respectively of either end of the rod to the circumference of the circle. If, further, we write

$$-n = p^2 = 4Aa/(A+a)^2, \quad . \quad . \quad . \quad . \quad (28)$$

and denote Legendre's complete third elliptic integral by the notation,

$$\left.\begin{aligned} \Pi_3 = \Pi_3(u, k, \tfrac{1}{2}\pi) &= \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1+n\sin^2\phi)(1-k^2\sin^2\phi)^{\frac{1}{2}}} \\ &= \int_0^{K} \frac{du}{1+n\sin^2 u}, \end{aligned}\right\} \quad . \quad . \quad . \quad (29)$$

we easily find that

$$I = 2\sqrt{\frac{a}{A}}\, kd\left[\frac{K-E}{k^2} + \frac{1-p^2}{p^2}(K-\Pi_3)\right]. \quad . \quad . \quad (30)$$

If Θ is the whole angle of the winding,

$$\tan \epsilon = A\Theta/d = 2\pi AN,$$

and the factor multiplying the terms in [ ] in the expression for $\tan \epsilon I$ in (23) may be written,

$$\Theta(A+a)pk = 2\pi N(A+a)pk, \quad . \quad . \quad . \quad (30.1)$$

where N is the number of turns in a length d of the helical coil. Formula (30) thus agrees with that given by Jones *.

The exact numerical evaluation of M by (30) making use of Legendre's tables of incomplete integrals of the first and second kinds (in terms of which the complete third elliptic integral may be expressed), is stated to be extremely tedious.

We may, however, replace (30) by an extremely convergent expression, making use of the A.G.M. scale

$$(a_0=1,\ b_0=k'), \quad \text{or} \quad A_0=R_1,\ B_0=R_2.$$

If we write

$$n = -p^2 = -1 + k'^2 \sin^2\psi_0, \quad . \quad . \quad . \quad . \quad (31)$$

and calculate successively $\psi_1, \psi_2, \ldots \psi_n \ldots$ from the recurrence-formula

$$\sin(2\psi_{n+1} - \psi_n) = (b_n/a_n) . \sin\psi_n, \quad . \quad . \quad . \quad (32)\dagger$$

it may be proved that

$$\Pi_3 - K = \frac{K\cos(2\psi_1-\psi_0)}{k'^2 \sin\psi_0 \cos\psi_0}\left[a_n(1-\sin\psi_n) - \sum_2(\psi_n, c_n)\right], \quad . \quad . \quad . \quad (33)$$

* Jones, J. V., Quoted in Scientific paper No. 169 of the Bureau of Standards (third edition, 1906) by E. B. Rosa and F. W. Grover, p. 99.

† The complete theory of the group of trigonometrical recurrence formulæ associated with the expression of elliptic functions and integrals in terms of the arithmetico-geometrical scales has been published by the writer in a booklet, 'On the Direct Numerical Calculation of Elliptic Functions and Integrals' (Cambridge University Press, 1924, cloth, pp. vi+42). Most of the formulæ of the present paper were obtained prior to the publication of this little volume, which, it was felt, should be available before such applications as those involved in mutual inductance calculations should be dealt with. In a separate paper of the present issue, Prof. Frederick W. Grover has dealt with numerical computations from the point of view of accuracy and economy of labour.

where

$$\left.\begin{aligned}\Sigma_2(\psi_n,\, c_n) = 2c_2\frac{\tan(2\psi_2-\psi_1)}{\cos(2\psi_3-\psi_2)}+6c_3\frac{\tan(2\psi_3-\psi_2)}{\cos(2\psi_4-\psi_3)}\\ +\ldots 2(2^{n+1}-1)c_{n+2}\frac{\tan(2\psi_{n+2}-\psi_{n+1})}{\cos(2\psi_{n+3}-\psi_{n+2})}+\ldots,\end{aligned}\right\} \quad . \quad . \quad . \quad (34)$$

Applying these formulæ we find

$$\left.\begin{aligned}\sin\psi_0 = \frac{(A-a)}{(A+a)}\frac{R_1}{R_2}, \quad \cos\psi_0 = \frac{2\sqrt{Aa}}{(A+a)}\frac{d}{R_2} = p\frac{d}{R_2},\\ \sin(2\psi_1-\psi_0) = \frac{A-a}{A+a}, \quad \cos(2\psi_1-\psi_0) = \frac{2\sqrt{Aa}}{A+a} = p,\end{aligned}\right\} \quad . \quad . \quad . \quad .(35)$$

while

$$2\sqrt{\frac{a}{A}}\,kd\,\frac{(1-p^2)}{p^2}\frac{\cos(2\psi_1-\psi_0)}{\sin\psi_0\cos\psi_0} = \frac{A^2-a^2}{A} \quad . \quad . \quad (36)$$

We note, also, that $K=\frac{1}{2}\pi/a_n$. We then find that (30) may be written

$$\left.\begin{aligned}I = \frac{A^2-a^2}{A}\frac{\frac{1}{2}\pi}{a_n}\Big[a_n\sin\psi_n+\Sigma_2(\psi_n,\, c_n)\\ +\frac{dR_1}{2(A^2-a^2)}(c_0{}^2+2c_1{}^2+4c_2{}^2+\ldots)\Big]-\tfrac{1}{2}\pi\frac{(A^2-a^2)}{A}.\end{aligned}\right\} \quad . \quad . \quad . \quad (37)^*$$

If desired we may refer the above formula to the A.G.M. scale

$$(A_0=R_1,\ B_0=R_2)$$

by writing $\qquad a_n=A_n/R_1, \qquad c_n=C_n/R_1$

in (37). There is little or no advantage in doing so, however, in this particular formula.

In calculating M from (23), we compute two expressions for I in which d takes the values d_2 and d_1 respectively. Since the last term of (37) is independent of d, it need not be calculated in evaluating (I_2-I_1).

From (26) it will be seen that $I=0$ when $d=0$. In this case (34) shows that

$$\psi_0=\tfrac{1}{2}\pi,$$

* An alternative formula in terms of the complementary A.G.M. scale, which in some cases it is advantageous to employ, has been obtained by Prof. F. W. Grover, and will be found in his paper in the present number.

in which circumstances it may be shown that

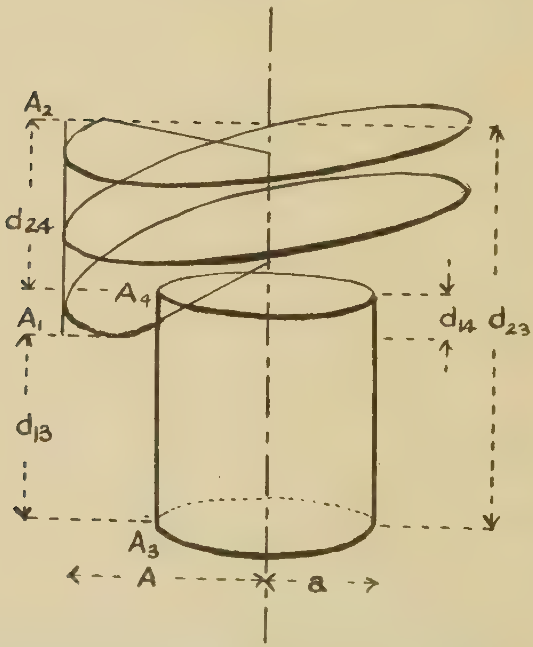
$$a_n = [a_n \sin\psi_n + \underset{2}{\Sigma}(\psi_n, c_n)]_{\psi_0=\frac{1}{2}\pi},$$

so that $I=0$ by (37), as it should.

Section 4.—*Mutual Inductance of a Cylindrical Current Sheet and Coaxial Helix.*

Let N′ be the number of turns per unit length of the cylinder, the windings being so close that the distribution

Fig. 5.



of current approximates to that in a cylindrical current-sheet.

The required result is obtained by integrating (23) with respect to d between the limits for which d takes the values corresponding to the positions A_3 and A_4 (fig. 5), the extremities of the generating line of the cylinder. It follows that

$$M = \tan\epsilon[I_{23} - I_{14} - I_{13} + I_{14}], \quad . \quad . \quad . \quad (38)$$

where each of the I's is an integral of the type

$$\mathrm{I}=2\mathrm{A}a^2\mathrm{N}'\int_0^\pi \frac{(d^2+r^2)^{\frac{1}{2}}}{r^2}\sin^2\theta.d\theta, \quad . \quad . \quad (39)$$

in which d takes the values d_{23}, d_{24} . . . . etc.

To reduce the integral (39), we proceed, as in Section 3, to write

$$\cos\tfrac{1}{2}\theta=\mathrm{sn}\,(u,k),$$

where, as before,

$$-n=4\mathrm{A}a/(\mathrm{A}+a)^2,$$

and

$$k^2=\frac{4\mathrm{A}a}{(\mathrm{A}+a)^2+d^2}, \qquad k'^2=\frac{\mathrm{R}_2{}^2}{\mathrm{R}_1{}^2}=\frac{(\mathrm{A}-a)^2+d^2}{(\mathrm{A}+a)^2+d^2}. \quad (40)$$

We easily find

$$r^2=(\mathrm{A}+a)^2(1+n\,\mathrm{sn}^2u), \qquad (d^2+r^2)^{\frac{1}{2}}=\mathrm{R}_1\mathrm{dn}\,u,$$

and

$$d\theta=2\mathrm{dn}\,u.du.$$

Thus,

$$\mathrm{I}=\frac{16\mathrm{A}a^2\mathrm{N}'\mathrm{R}_1}{(\mathrm{A}+a)^2}\int_0^{\mathrm{K}}\frac{\mathrm{sn}^2u.\mathrm{cn}^2u.\mathrm{dn}^2u}{1+n\,\mathrm{sn}^2u}.du \quad . \quad (41)$$

$$=-4n\mathrm{N}'a\mathrm{R}_1\int_0^{\mathrm{K}}\frac{s^2(1-s^2)(1-k^2s^2)}{1+ns^2}du,$$

where, for brevity, we have written s for sn u, c for cn u, and d for dn u.

We may write the integrand in the form

$$\frac{s^2(1-s^2)(1-k^2s^2)}{1+ns^2}=\mathrm{A}s^4+\mathrm{B}s^2+\mathrm{C}+\frac{\mathrm{D}}{1+ns^2}. \quad (42)$$

We easily find

$$\int_0^{\mathrm{K}}s^2du=\frac{1}{k^2}(\mathrm{K}-\mathrm{E}), \qquad \int_0^{\mathrm{K}}s^4du=\frac{2(1+k^2)}{3k^2}\frac{(\mathrm{K}-\mathrm{E})}{k^2}-\frac{\mathrm{K}}{3k^2},$$

and

$$\int_0^{\mathrm{K}}\frac{du}{1+us^2}=\Pi_3(n,k,\tfrac{1}{2}\pi),$$

where $\Pi_3(n, k, \frac{1}{2}\pi)$ is Legendre's complete third elliptic integral.

We thus have,

$$\int_0^K \frac{s^2c^2d^2.du}{1+ns^2}$$

$$=\left\{\frac{2A(1+k^2)}{3k^2}+B\right\}\frac{(K-E)}{k^2}-\frac{AK}{3k^2}+CK+D\Pi_3.$$

We easily find

$$D=-C=-\frac{1}{n}\left(1+\frac{1}{n}\right)\left(1+\frac{k^2}{n}\right)=-\frac{1}{n^2}\frac{(A-a)^2}{(A+a)^2}\cdot\frac{d^2}{R_1^2}.$$

Furthermore,

$$A=k^2/n, \quad B=-(1+k^2)/n-k^2/n^2,$$

and

$$\frac{2A(1+k^2)}{3k^2}+B=-\left\{\frac{1}{3}\frac{(1+k^2)}{n}+\frac{k^2}{n^2}\right\}.$$

Remembering that

$$\tan\epsilon=A\,\Theta/d=2\pi AN,$$

where N is the number of turns in unit distance of the helix, we may write,

$$M=2\pi AN\,[I_{23}-I_{24}-I_{13}+I_{14}], \quad . \quad . \quad . \quad (43)$$

where

$$I=4N'aR_1\left[\left\{\tfrac{1}{3}(1+k^2)+\frac{k^2}{n}\right\}\frac{(K-E)}{k^2}\right.$$

$$\left.+\tfrac{1}{3}K+\frac{(A-a)^2}{(A+a)^2}\frac{d^2}{nR_1^2}(\Pi_3-K)\right]. \quad (44)$$

In the expression (43), each term I is calculated for each of the values

$$d=d_{23},\ d_{24},\ d_{13},\ \text{and}\ d_{14},$$

as indicated in fig. 5.

The computation for (K−E), K, and (Π_3-K) in terms of the A.G.M. scale proceeds as in Section 3.

Section V.—*Self-Inductance of Cylindrical Current Sheet.*

Consider first of all the mutual inductance of co-axial and concentric cylindrical current-sheets of lengths l and radii (A, a).

In applying (38), we have

$$d_{23}=l,\ d_{13}=0,\ d_{24}=0,\ d_{14}=l.$$

Thus, making $N=N'$, and ultimating $A=a$,

$$M=4\pi AN\{I(l)-I(0)\},$$

and gives in the limit, when $A\to a$,

$$L=4\pi aN\{I(l)-I(0)\}_{A=a}. \quad . \quad . \quad . \quad . \quad (45)$$

In these circumstances the last term in (44) disappears, and since $-n=1$,

$$[I(l)]_{A=a}=4NaR_1\left[(\tfrac{1}{3}-\tfrac{2}{3}k^2)\frac{(K-E)}{k^2}+\tfrac{1}{3}K\right], \quad . \quad (46)$$

where we now have, according to (40), with $d=l$,

$$k^2=\frac{4a^2}{4a^2+l^2}, \qquad k'^2=\frac{l^2}{4a^2+l^2}. \quad . \quad . \quad (47)$$

Since $R_1{}^2=(4a^2+l^2)$, we may write

$$[I(l)]_{A=a}=\frac{4NaR_1}{3}\left[K+\left(\frac{l^2}{4a^2}-1\right)(K-E)\right], \quad . \quad (48)$$

which is easily expressed in terms of the A.G.M. scale, commencing with

$$A_0=l, \qquad B_0=R_1=(4a^2+l^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad (49)$$

We may write (48) in the form

$$[I(l)]_{A=a}=\frac{4NaR_1}{3}\left[\frac{l^2}{4a^2}K-\left(\frac{l^2}{4a^2}-1\right)E\right].$$

When $l/a \to 0$, $R_1 \to 2a$, $k \to 1$, $E(k) \to 1$, and

$$\operatorname*{Lt}_{l/a\to 0}[(l^2/4a^2)K(k)]\to 0, \qquad \operatorname*{Lt}_{l/a\to 0}[I(l)]_{A=a}\to\tfrac{8}{3}Na^2.$$

We thus have

$$L=\frac{16\pi N^2a^2}{3}[(4a^2+l^2)^{\frac{1}{2}}\{K+(\tfrac{1}{4}l^2/a^2-1)(K-E)\}-2a]. \quad . \quad . \quad . \quad (50)$$

When (l/a) becomes small, we may with advantage use the complementary A.G.M. scale as in Section 2.

The reader will have no difficulty in adapting (50) to A.G.M. computation, which proceeds after the manner of the preceding sections.

As a test of formula (50), consider the case of a very long solenoid such that $l/a \to \infty$ and, in consequence, $k \to 0$.

From the series expansions for K and E,

$$K=\tfrac{1}{2}\pi\{1+\tfrac{1}{4}k^2+\ldots\}, \qquad E=\tfrac{1}{2}\pi\{1-\tfrac{1}{4}k^2+\ldots\}$$

$$K-E\sim\tfrac{1}{4}\pi k^2=\tfrac{1}{4}\pi\,.\,4a^2/(4a^2+l^2).$$

Thus, in (6),

$$\left(\frac{l^2}{4a^2}-1\right)(\mathrm{K}-\mathrm{E}) \sim \frac{l^2-4a^2}{l^2+4a^2}\cdot\frac{\pi}{4} \sim \frac{\pi}{4}, \text{ when } l/a \to \infty.$$

Thus

$$\underset{l/a \to \infty}{\mathrm{Lt}} (\mathrm{L}/l) \sim \tfrac{16}{3}\pi \mathrm{N}^2 a^2 [(4a^2+l^2)^{\frac{1}{2}}(\tfrac{1}{2}\pi+\tfrac{1}{4}\pi)-2a]$$

$$\sim \tfrac{16}{3}\pi \mathrm{N}^2 a^2 . \tfrac{3}{4}\pi = 4\pi^2 \mathrm{N}^2 a^2. \quad . \quad . \quad . \quad . \quad . \quad (51)$$

It is easy to see that this is the correct value, since, in such a case, for unit current, $\mathrm{H} = 4\pi\mathrm{N}$, and equating

$$\tfrac{1}{2}\mathrm{L}i^2 = \frac{1}{8\pi}\int \mathrm{H}^2 d(vol),$$

$$\tfrac{1}{2}\mathrm{L} = \frac{1}{8\pi}\mathrm{H}^2 . \pi a^2 l, \quad \text{or} \quad \mathrm{L}/l = \tfrac{1}{4}\mathrm{H}^2 a^2 = 4\pi^2 \mathrm{N}^2 a^2,$$

in agreement with the special case (51).

Summary and Conclusions.

1. From Ampère's expression for the contribution of a current element to the magnetostatic field of an entire circuit, it is possible to obtain a new and useful expression for the mutual inductance in the following cases:—

(i.) Maxwell's two formulæ for co-axial circles.
(ii.) Circle and co-axial helix.
(iii.) Cylindrical current sheet and co-axial helix.
(iv.) Self-inductance of cylindrical current sheet.

2. The use of the arithmetico-geometrical scales used by the writer in 1921 to obtain highly convergent expressions for the mutual inductance of co-axial circles, has now been extended, by the use of trigonometrical recurrence formulæ, to include useful computational forms for the direct calculation of M, without reference to tables of elliptic integrals, for the above-mentioned circuits forming part of absolute electrical standards equipment. Such calculations are now required to an accuracy represented by arithmetical computations to at least seven significant figures.

Metis Beach, Quebec.
September 9th, 1932.

XC. *On the Use of Arithmetico-geometric Mean Series for the Calculation of Elliptic Integrals, with Special Reference to the Calculation of Induction.* By FREDERICK W. GROVER, *Professor of Electrical Engineering, Union College, Schenectady, N.Y., U.S.A.**.

THE absolute formulæ for the calculation of the self and mutual inductance of circular coils involve elliptic integrals. In the simpler cases, such as those of the mutual inductance of coaxial circles and the self inductance of cylindrical coils or current sheets, the complete elliptic integrals of the first and second kinds alone appear. In the important case of the mutual inductance of a solenoid and coaxial circle (Lorenz apparatus and Campbell form of mutual inductance standard) and in the case of the mutual inductance of coaxial solenoids, the complete elliptic integral of the third kind enters also.

Tables (1) of the complete elliptic integrals of the first and second kinds are quite accessible and enable values of sufficient accuracy for most practical purposes to be readily interpolated. For the complete elliptic integral of the third kind it is customary to make use of a relation which involves, not only the complete elliptic integrals of the first and second kinds, but the incomplete integrals also.

For the incomplete elliptic integrals of the first and second kinds the very extensive tables of Legendre, recently made more widely available by a new photographic reprint by Emde (2), are generally used.

The present-day accuracy of absolute electrical measurements demands at least a seven-figure accuracy in the calculated constant of the standards of mutual inductance employed, and to attain this accuracy of calculation the interpolations necessary in the Legendre tables, voluminous though those tables are, are time-consuming and tedious.

However, in such exacting work, it is not difficult to avoid the use of tables of elliptic integrals altogether, and to make the calculation directly by formulæ based on the method of the arithmetico-geometric mean of Gauss (3). This method has been developed at length by L. V. King (4), who has given formulæ, not only for the

* Communicated by the Author.

calculation of elliptic integrals of all three kinds, but for the calculation of inductance in those cases especially important for absolute measurements.

The present paper has for its object the illustration of the use of these formulæ for numerical calculation. Seven-place logarithms have been employed, since they give an accuracy sufficient for all except unusual cases. For the latter, eight-place logarithms, such as those of Bauschinger and Peters [5] should suffice, or a calculating machine with a table of natural trigonometric functions such as that of Roussilhe and Brandicourt [6] may be employed.

The agreement of the results found by arithmetico-geometric mean series formulæ (hereafter referred to as A.G.M. series formulæ) with those resulting from the use of Legendre's tables is also illustrated, and an indication given of the relative amounts of time required for calculations by the two methods.

The Gauss Arithmetico-geometric Mean Series.

This may be based upon any two positive numbers a_0 and b_0, of which the former is the greater, and the successive terms are calculated according to the following scheme :—

$$
\begin{array}{lll}
a_1 = \tfrac{1}{2}(a_0+b_0), & b_1 = \sqrt{a_0 b_0}, & c_1 = \tfrac{1}{2}(a_0-b_0), \\
a_2 = \tfrac{1}{2}(a_1+b_1), & b_2 = \sqrt{a_1 b_1}, & c_2 = \tfrac{1}{2}(a_1-b_1), \\
a_3 = \tfrac{1}{2}(a_2+b_2). & b_3 = \sqrt{a_2 b_2}. & c_3 = \tfrac{1}{2}(a_2-b_2).
\end{array}
$$

The quantity c_0 is found by the relation $c_0^2 = (a_0^2 - b_0^2)$.

The successive arithmetic means a_m and geometric means b_m approach each other very rapidly in value, and converge to a definite value a_n, which may be called the convergent of the series. The successive values c_m rapidly approach zero as a limit. To make use of the A.G.M. series for the calculation of elliptic integrals, it is convenient to chose a_0 equal to unity. Two A.G.M. scales are then possible, according as b_0 is chosen equal to the complementary modulus $k' = \sqrt{1-k^2}$ of the elliptic integral, or to the modulus k. Since the formulæ are, in general, simpler in the former case, the latter series will be regarded as the *complementary* series.

Formulæ for the Elliptic Integrals K and E of the First and Second Kinds.

The integrals are supposed to have a modulus k.

With the A.G.M. scale $a_0=1$, $b_0=k'$, and therefore $c_0=k$, the formulæ for the elliptic integrals in terms of the convergent a_n and quantities c_m are

$$\mathrm{K}=\mathrm{F}\left(k,\frac{\pi}{2}\right)=\frac{\pi}{2}\left(\frac{1}{a_n}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\mathrm{E}}{\mathrm{K}}=\frac{\mathrm{E}\left(k,\frac{\pi}{2}\right)}{\mathrm{F}\left(k,\frac{\pi}{2}\right)}=1-\tfrac{1}{2}(k^2+2c_1{}^2+4c_2{}^2+\ldots+2^m c_m^2+\ldots) \quad . \quad (2)$$

The quantity $\frac{\mathrm{K}-\mathrm{E}}{\mathrm{K}}$, which is of frequent occurrence in the applications of elliptic integrals, is obtained from the simple formula

$$\frac{\mathrm{K}-\mathrm{E}}{\mathrm{K}}=\tfrac{1}{2}(k^2+2c_1{}^2+4c_2{}^2+\ldots+2^m c_m^2+\ldots.). \quad . \quad (3)$$

To illustrate these formulæ in the rather unfavourable case $k=0{\cdot}6$, the complementary modulus is found to be $k'=0{\cdot}8$, and the series (1, 0·8) is formed.

$\log a_0=0$,	$a_0=1$,	
,, $b_0=\bar{1}{\cdot}9030900$,	$b_0=0{\cdot}8$,	
$\log b_1=\bar{1}{\cdot}9515450$,	$a_1=0{\cdot}9$,	$c_1=0{\cdot}1$,
,, $a_1=\bar{1}{\cdot}9542425$,	$b_1=0{\cdot}8944273$,	
$\log b_2=\bar{1}{\cdot}9528937$,	$a_2=0{\cdot}8972136$,	$c_2=\frac{1}{2}(0{\cdot}0055727)$,
,, $a_2=\bar{1}{\cdot}9528958$,	$b_2=0{\cdot}8792092$,	$=0{\cdot}0027863_5$,
$\log b_3=\bar{1}{\cdot}9528947_5$,	$a_3=0{\cdot}8972114$,	$c_3=\frac{1}{2}(4{\cdot}4\times10^{-6})$,
,, $a_3=\bar{1}{\cdot}9528947_5$,	$b_3=0{\cdot}8972114$,	$=2{\cdot}2\times10^{-6}$,
$\log a_n=\bar{1}{\cdot}9528947_5$,	$a_n=0{\cdot}8972114$,	c_4 negligible.

Thus the value of K (0·6) is

$$\frac{\pi}{2}\left(\frac{1}{0{\cdot}8972114}\right)=1{\cdot}750754.$$

$\log\frac{\pi}{2}=0{\cdot}1961199$,

,, $a_n=\bar{1}{\cdot}9528947_5$,

$\log \mathrm{K}=0{\cdot}2432251_5$.

This value checks Nagaoka [(1)] and Sakurai's tabular value exactly, and agrees with the value interpolated from Legendre's tables.

It will be noticed that, starting with the values of a_0 and b_0 and their logarithms, the successive values of the a_m are derived by taking the arithmetic means of the values of a_{m-1} and b_{m-1}, and the successive values of log b_m are the arithmetic means of the values of log a_{m-1} and log b_{m-1}. The values of the b_m for use in calculating the a_{m+1} are taken from their logarithms, while the logarithms of the a_m have to be found for the calculation of the log b_{m+1}. The calculation is rapid and interesting.

To calculate E (0·6) it is necessary to use the values of the c_m in the foregoing calculation. Since, in formula (2), the calculation of E is made to depend upon the value of K, the value of a_n is involved implicitly.

$$\begin{aligned} k^2 &= 0{\cdot}36, \\ 2c_1^2 &= 0{\cdot}02, \\ 4c_2^2 &= 0{\cdot}00003105_5 \\ \tfrac{1}{2}\,\text{Sum} &= 0{\cdot}19001553, \\ \frac{\text{E}}{\text{K}} &= 0{\cdot}80998447. \end{aligned} \qquad \begin{aligned} \log\frac{\text{E}}{\text{K}} &= \bar{1}{\cdot}9084767, \\ \text{,,}\ \ \text{K} &= 0{\cdot}2432251_5, \\ \log \text{E} &= 0{\cdot}1517018_5\,; \\ \text{E} &= 1{\cdot}418084. \end{aligned}$$

Nagaoka's table gives a value one unit smaller in the last place, and the same is true of the value interpolated from Legendre's tables. Using a calculating machine the value 1·4180834 is found, agreeing with Legendre's value.

The extreme rapidity of convergence of the A.G.M. series is made manifest in these calculations, and the convergence is not unsatisfactory, even when k' is as small as 0·1. However, in such cases, and those in general for which interpolation is difficult in a table of values of K, the formulæ based on the complementary A.G.M. scale are more convenient.

Complementary Scale (1, k).

Placing $a_0'=1$, $b_0'=k$, and therefore $c_0'=k'$, the A.G.M. series is formed and its convergent a_n' found in the same manner as in the preceding case. The formulæ for the elliptic integrals in this case are

$$a_n'\text{K} = \tfrac{1}{2}\log\frac{4a_1'}{c_1'} - \sum_{m=1} (\tfrac{1}{2})^m \log\frac{a_m'}{a_{m+1}'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

$$= \tfrac{1}{2}\log\frac{4'a_2'}{c_1'} - \frac{1}{4}\log\frac{a_2'}{a_3'} - \frac{1}{8}\log\frac{a_3'}{a_4'} . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

$$\text{E} = \tfrac{1}{2}(k'^2 + 2c_1^2 + 4c_2'^2 + \ldots + 2^m a_m'^2 + \ldots) + a_n' . \quad . \quad . \quad . \qquad (6)$$

These formulæ are convenient when k is not far different from unity, since, in such cases, only the first term of the series is often required, or at most the first two terms. A difficulty is, however, met when k is very close to unity, in that the value of c_1' is small and difficult to obtain with sufficient accuracy. This obstacle may be overcome by noting that

$$\log \frac{4a_1'}{c_1'} = \log \frac{4(1+k)^2}{k'^2},$$

so that formula (5) may be written

$$a_n' \mathrm{K} = \frac{1}{2}\log \frac{4(1+k)^2}{k'^2} - \frac{1}{2}\log \frac{a_1'}{a_2'} - \frac{1}{4}\log \frac{a_2'}{a_3'} \dots$$

$$= \frac{1}{2}\log \frac{2(1+k)(1+\sqrt{k})^2}{k'^2} - \frac{1}{4}\log \frac{a_2'}{a_3} \dots \quad . \quad (7)$$

The natural logarithms are conveniently obtained from the common logarithms by making use of the multiplication table for the factor 2·302585 . . usually given in books of logarithmic tables. It is evident that this conversion has to be made once only, *i. e.*, after the common logarithms have been combined as indicated in the formulæ. The complementary modulus can usually be obtained directly from the geometric data of the problem.

To illustrate the use of the formulæ (4) to (7) we will again obtain the value of the integrals for the modulus 0·6.

Forming the A.G.M. series (1, 0·6)

$\log a_0' = 0,$	$a_0' = 1,$	
,, $b_0' = \bar{1}\cdot7781513,$	$b_0' = 0\cdot6,$	
$\log b_1' = \bar{1}\cdot8890756_5,$	$a_1' = 0\cdot8,$	$c_1 = 0\cdot2,$
,, $a_1' = \bar{1}\cdot9030900,$	$b_1' = 0\cdot7745967$	
$\log b_2' = \bar{1}\cdot8960828,$	$a_2' = 0\cdot7872983,$	$c_2 = 0\cdot0127016,$
,, $a_2' = \bar{1}\cdot8961393_5,$	$b_2' = 0\cdot7871958,$	
$\log b_3' = \bar{1}\cdot8961111,$	$a_3' = 0\cdot7872470,$	$c_3 = 5\cdot125 \times 10^{-5}.$
,, $a_3' = \bar{1}\cdot8961111,$	$b_3' = 0\cdot7872470,$	
$\log a_n' = \bar{1}\cdot8961111.$	$a_n' = 0\cdot7872470.$	

Since the value of c_1' is given exactly formula (5) may be used.

$$\begin{aligned}
\log 4 &= 0{\cdot}6020600\\
,,\ a_2' &= \bar{1}{\cdot}8961393_5,\\
\text{Sum} &= 0{\cdot}4981993_5,\\
\log c_1' &= \bar{1}{\cdot}3010300,\\
\log_{10}\frac{4a_2'}{c_1'} &= 1{\cdot}1971693_5\\
\log_{10}\frac{a_2'}{a_3'} &= 0{\cdot}00002825,\\
\log_{10} a_n'\,\mathrm{K} &= 0{\cdot}1393362,\\
\log a_n' &= \bar{1}{\cdot}8961111,\\
\log \mathrm{K} &= 0{\cdot}2432251.
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}\log_{10}\frac{4a_2'}{c_1'} &= 0{\cdot}5985847\\
\tfrac{1}{4}\log_{10}\frac{a_2'}{a_3'} &= 0{\cdot}0000071\\
\text{Diff.} &= 0{\cdot}5985776.\\
\text{Multiplying by } &2{\cdot}3025851,\\
&1{\cdot}3585252,\\
&195720,\\
&1773,\\
&14,\\
a_n'\,\mathrm{K} &= 1{\cdot}3782759.
\end{aligned}$$

This checks the value of log K found before to the last place of logarithms.

To illustrate the use of formula (7) there is found

$$\begin{aligned}
1+k &= 1+b_0' = 1{\cdot}6,\\
1+\sqrt{k} &= 1+b_1' = 1{\cdot}7745967,\\
\therefore\ a_n'\mathrm{K} &= 1{\cdot}3782761\\
\log_{10} a^{n\prime}\mathrm{K} &= 0{\cdot}1393362,\\
\log a_n' &= \bar{1}{\cdot}8961111,\\
\therefore\ \log \mathrm{K} &= 0{\cdot}2432251,
\end{aligned}$$

as before.

$$\begin{aligned}
\log_{10}(1+k) &= 0{\cdot}2041200,\\
2\log_{10}(1+\sqrt{k}) &= 0{\cdot}4981996\\
\log 2 &= 0{\cdot}3010300\\
\text{Sum} &= 1{\cdot}0033496,\\
\log k'^2 &= \bar{1}{\cdot}8061800\\
\text{Diff.} &= 1{\cdot}1971696\\
\tfrac{1}{2}\ \text{Diff.} &= 0{\cdot}5985848\ ;\\
\tfrac{1}{4}\log_{10}\frac{a_2'}{a_3'} &= 0{\cdot}0000071\\
\text{Diff.} &= 0{\cdot}5985777.\\
\text{Times } 2{\cdot}302585_1 &= 1{\cdot}3782761.
\end{aligned}$$

Calculating E(0·6) by formula (6)

$$\begin{aligned}
k'^2 &= 0{\cdot}64\\
2c_1'^2 &= 0{\cdot}08\\
4c_2'^2 &= 0{\cdot}0006453_2\\
8c_3'^2 &= 0{\cdot}0000000_2\\
\text{Sum} &= 0{\cdot}7206453_4,\\
\tfrac{1}{2}\ \text{sum} &= 0{\cdot}3603227.
\end{aligned}$$

$$\begin{aligned}
\log(\tfrac{1}{2}\ \text{sum}) &= \bar{1}{\cdot}5566917,\\
\log \mathrm{K} &= 0{\cdot}2432251,\\
&\ \bar{1}{\cdot}7999168\ ;\\
\therefore\ \text{First term} &= 0{\cdot}6308367,\\
a_n' &= 0{\cdot}7872470,\\
\therefore\ \mathrm{E} &= 1{\cdot}4180837,
\end{aligned}$$

which is the same value as was found by formula (2), carried to one more place, and as near the correct value as the accuracy of 7-place logarithms will allow.

Calculation of the Incomplete Elliptic Integrals of the First and Second Kinds.

These will be written as F(k, ϕ) and E(k, ϕ), respectively, k being the modulus and ϕ the amplitude To calculate these incomplete elliptic integrals it is necessary to form an A.G.M. scale, as in the case of the complete integrals, and, in addition, to calculate the limiting value oi an angle from the given amplitude, using a trigonometric recurrence formula.

Formulæ based on the A.G.M. *Scale* $(1, k')$.

Having formed the scale as usual, the values of the $\dfrac{b_m}{a_m}$ are available. Writing ϕ in place of ψ_0, repeated use is made of the recurrence formula

$$\tan(\psi_{m+1} - \psi_m) = \frac{b_m}{a_m} \tan \psi_m \quad . \quad . \quad . \quad (8)$$

The differences $(\psi_{m+1} - \psi_m)$ rapidly approach a limit as $\dfrac{b_m}{a_m}$ approaches unity, so that the ratio $\dfrac{\psi_m}{2^m}$ rapidly approaches a limit which will be designated as $\dfrac{\psi_n}{2^n}$.

Then

$$F(k, \phi) = \frac{1}{a_n}\left(\frac{\psi_n}{2^n}\right) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$E(k, \phi) = [1 - \tfrac{1}{2}(k^2 + 2c_1{}^2 + 4c_2{}^2 + \ldots + 2^m c_m^2)] \,.\, F(k, \phi)$$
$$+ [c_1 \sin \psi_1 + c_2 \sin \psi_2 + \ldots] \quad . \quad . \quad (10)$$

These formulæ will be used to calculate the elliptic integrals with a complementary modulus $k' = 0{\cdot}9299812$ and whose amplitude is given by $\log \sin \phi = \bar{1}{\cdot}7977938$, that is, $\phi = 38° \; 35' \; 06{\cdot}''25$. (These integrals occurred in a certain [7] inductance calculation.)

The formation of the A.G.M. series yields the data

$$a_2 = b_2 = a_n = 0{\cdot}9646729, \qquad \log \frac{b_0}{a_0} = \bar{1}{\cdot}9684741,$$

$$\log a_n = \bar{1}{\cdot}9843801. \qquad \log \frac{b_1}{a_1} = \bar{1}{\cdot}9997140.$$

The calculation by the recurrence formulæ is as follows :—

$\log \sin \psi_0 = \bar{1}{\cdot}7977938,$	
$\log \tan \psi_0 = \bar{1}{\cdot}9065872,$	
,, $b_0/a_0 = \bar{1}{\cdot}9684741_5,$	$\psi_0 = 38° \; 35' \; 06''{\cdot}25,$
$\log \tan(\psi_1 - \psi_0) = \bar{1}{\cdot}8750613,$	$\psi_1 - \psi_0 = 36° \; 52' \; 11''{\cdot}65,$
$\log \tan \psi_1 = 0{\cdot}5953796,$	$\psi_1 = 75° \; 45' \; 17''{\cdot}90,$
,, $b_1/a_1 = \bar{1}{\cdot}9997140,$	$\dfrac{\psi_1}{2} = 37° \; 52' \; 38''{\cdot}95,$
$\log \tan(\psi_2 - \psi_1) = 0{\cdot}5950936,$	$\psi_2 - \psi_1 = 75° \; 44' \; 45''{\cdot}50,$
	$\psi_2 = 151° \; 30' \; 03'{\cdot}40,$
	$\dfrac{\psi_2}{4} = 37° \; 52' \; 30''{\cdot}85,$
$\log \dfrac{\psi_n}{2_n} = \bar{1}{\cdot}82023265 ;$	$\therefore \; \dfrac{\psi_n}{2_n} = 37° \; 52' \; 30''{\cdot}85,$
,, $a_n = \bar{1}{\cdot}9843801,$	$= 0{\cdot}6610475$ radians ;
$\log F(k, \phi) = \bar{1}{\cdot}8358525_5 ;$	$\therefore \; F(k, \phi) = 0{\cdot}6852556.$

Legendre's tables yield after interpolation for both modulus and amplitude, the value 0·68525574. By the calculating machine formula (9) gives 0·68525561.

For the calculation of $E(k, \phi)$ by formula (10) we need in addition the values of the c_m in the A.G.M. scale and the sines of the ψ_m in the calculation by the recurrence formula.

$$
\begin{array}{rl@{\qquad}rl}
\log k = & \bar{1}\cdot5653841, & \log \sin \psi_1 = & \bar{1}\cdot9864367, \\
c_1 = & 0\cdot0350094, & ,, \quad \sin \psi_2 = & \bar{1}\cdot6786616, \\
c_2 = & \underline{0\cdot0003176_5}, & & \\
k^2 = & 0\cdot1351351, & & \\
2c_1^2 = & 0\cdot0024513, & & \\
4c_2^2 = & \underline{0\cdot0000004}, & & \\
\tfrac{1}{2}\text{ sum} = & 0\cdot0687934, & & \\
1-\tfrac{1}{2}\text{ sum} = & 0\cdot9312066, & 0\cdot0339329_4 = & c_1 \sin \psi_1, \\
\log = & \bar{1}\cdot9690461, & 0\cdot0001515_7 = & c_2 \sin \psi_2, \\
& & \underline{0\cdot0340845} = & \text{sum}, \\
\log F(k, \phi) = & \underline{\bar{1}\cdot8358525}, & & \\
& \bar{1}\cdot8048986. & = & \underline{0\cdot6381144}, \\
& & 0\cdot6721989 = & E(k, \phi).
\end{array}
$$

Legendre's table gives the value 0·67219891.

Use of the A.G.M. *scale* $(1, k)$.

Having formed this scale in the usual way, the results are used to calculate the limiting angle ψ_n' from the recursion formula

$$\sin (2\psi'_{m+1} - \psi'_m) = \frac{b_m}{a_m} \sin \psi'_m, \quad . \quad . \quad . \quad (11)$$

starting with $\psi_0' = \phi$. Then

$$F(k, \phi) = \frac{1}{a_n'} \log \tan\left(\frac{\pi}{4} + \frac{\psi_n'}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$E(k, \phi) = \tfrac{1}{2}(k'^2 + 2c_1'^2 + 4c_2'^2 + \ldots + 2^m c_m'^2 + \ldots).F(k, \phi) \\ + a_n' \sin \psi_n' + \Sigma. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which

$$\Sigma = 2c_2' \frac{\tan (2\psi_2' - \psi_1')}{\cos (2\psi_3' - \psi_2')} + 6c_3' \frac{\tan (2\psi_3' - \psi_2')}{\cos (2\psi_4' - \psi_3')} + \ldots \\ + 2(2^{m+1} - 1)c'_{m+2} \frac{\tan (2\psi'_{m+2} - \psi'_{m+1})}{\cos (2\psi'_{m+3} - \psi'_{m+2})} + \ldots . \quad (14)$$

These formulæ will be illustrated by the calculation of the same incomplete elliptic integrals as were obtained from the formulæ for the other A.G.M. scale.

Placing b_0' equal to the modulus 0·3676073, the convergent of the A.G.M. series is $\log a_n' = 1·8092043$, and

$$\log \frac{b_0'}{a_0'} = \bar{1}·5653841, \qquad \log \frac{b_3'}{a_3'} = \bar{1}·9999998, \qquad c_2' = 0·0387487,$$
$$\text{,,} \ \frac{b_1'}{a_1'} = \bar{1}·9477606, \qquad c_1' = 0·3161963_5, \qquad c_3' = 0·0005825,$$
$$\text{,,} \ \frac{b_2'}{a_2'} = \bar{1}·9992149. \qquad c_4' = 1·5 \times 10^{-7}.$$

The calculation of ψ_n' is made starting from the same angle as in the previous example.

$$\begin{aligned}
\psi_0' &= 38° \ 53' \ 06''·25, \\
2\psi_1' - \psi_0' &= 13° \ 20' \ 32''·51, \\
\psi_1' &= 26° \ 06' \ 49''·38, \\
2\psi_2' - \psi_1' &= 22° \ 58' \ 16''·73, \\
\psi_2' &= 24° \ 32' \ 33''·06, \\
2\psi_3' - \psi_2' &= 24° \ 29' \ 42''·98, \\
\psi_3' &= 24° \ 31' \ 08''·02, \\
2\psi_4' - \psi_3' &= 24° \ 31' \ 07''·98 \,; \\
\psi_n' = \psi_4' &= 24° \ 31' \ 08''·00, \\
\frac{\pi}{4} + \frac{\psi_n'}{2} &= 57° \ 15' \ 34''·00,
\end{aligned}$$

$$\begin{aligned}
\log_{10} \tan\left(\frac{\pi}{4} + \frac{\psi_n'}{2}\right) &= 0·1917967, \\
\log \tan\left(\frac{\pi}{4} + \frac{\psi_n'}{2}\right) &= 0·4416282, \\
\log \log &= \bar{1}·6450568, \\
\log a_n' &= \bar{1}·8092043, \\
\log F(k, \phi) &= \bar{1}·8358525 \,; \\
\therefore F(k, \phi) &= 0·6852556,
\end{aligned}$$

agreeing with the value found by formula (9).

To find $E(k, \phi)$ requires the calculation of three terms. The first two are as follows :—

$$\begin{aligned}
k'^2 &= 0·8648650, \\
2c_1'^2 &= 0·1999602, \\
4c_2'^2 &= 0·0060058, \\
8c_3'^2 &= 0·0000027, \\
\tfrac{1}{2} \text{ sum} &= 0·5354168.
\end{aligned}$$

Multiplying by $F(k, \phi)$,
First term $= 0·3668972$.

$$\begin{aligned}
\log \sin \psi_n' &= \bar{1}·6180410, \\
\log a_n' &= \bar{1}·8092043, \\
& \ \ \bar{1}·4272453,
\end{aligned}$$

Second term $= 0·2674517$.

The Σ term is the smallest. Three terms of the series (14) suffice.

$$\begin{array}{r}
0·03609878 \\
·00174999 \\
·00000105 \\
\hline
0·03785002 = \Sigma.
\end{array}$$

Thus the elliptic integral is given by the sum

$$\begin{array}{l}
0·3668972 \\
0·2674517 \\
0·0378500 \\
\hline
0·6721989 = E(k, \phi).
\end{array}$$

It is evident that the formulæ for $F(k, \phi)$ and $E(k, \phi)$, using the scale $(1, k)$, are favourable only when the modulus k is nearly unity, so that the present example is a severe test.

Calculation of the Complete Elliptic Integral of the Third Kind.

The complete elliptic integral of the third kind may be written as

$$\Pi_3(n, k)=\int_0^{\frac{\pi}{2}}\frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-k^2\sin^2\phi}}.$$

King [4] gives methods and formulæ for the calculation in terms of A.G.M. series of not only the complete, but also the incomplete elliptic integral of the third kind, but in the usual practical applications, we are concerned with the complete integral only. King's formulæ apply to four cases, depending upon the value of the parameter n. We will confine our attention here to the case where the value of n lies between $-k^2$ and -1. This condition is satisfied by the integrals which enter into the inductance formulæ cited below.

King [4] has developed two closely related formulæ for the integral based on the scale $(1, k')$, and the writer the corresponding formulæ depending on the complementary scale.

Formulæ based on the A.G.M. *scale* $(1, k')$.

Placing $n=-1+k'^2\sin^2\theta'=-c^2$, and writing the defining equation $c'^2=1-c^2$, it follows that $\frac{c'}{k'}=\sin\theta'$. Having calculated the A.G.M. scale in the usual way the recurrence formula (11) is applied, only this time θ' is placed equal to ψ_0'. Writing the abbreviation Π_3 for $\Pi_3(n, k)$, King's formula reads

$$\frac{\Pi_3-K}{K}=\frac{2\cos(2\psi_1'-\psi_0')}{k'^2\sin 2\psi_0'}[a_n(1-\sin\psi_n')-\Sigma]$$

$$=\frac{c}{c'\sqrt{c^2-k^2}}[a_n(1-\sin\psi_n')-\Sigma]. \quad . \quad . \quad (15)$$

where Σ is calculated by (14).

In King's second formula

$$n=-c^2=-\frac{k^2}{1-k'^2\sin^2\theta}, \text{ so that } \sin^2\theta=\frac{c^2-k^2}{c^2k'^2}.$$

Placing θ for ψ_0' in the recurrence formula (11) and calculating the limiting angle ψ_n',

$$\frac{\Pi_3}{\mathrm{K}}=\frac{2\cos(2\psi_1'-\psi_0')}{k'^2\sin 2\psi_0'}[a_n\sin\psi_n'+\Sigma]$$

$$=\frac{c}{c'\sqrt{c^2-k^2}}[a_n\sin\psi_n'+\Sigma], \quad . \quad . \quad . \quad (16)$$

where Σ is calculated by (14). It must be remembered, however, that the angles ψ_n' and the quantity Σ in this formula are not the same as the corresponding quantities in (15).

Formulæ based on the A.G.M. *scale* (1, k).

In the complementary formulæ which follow, the angles θ' and θ have the same definitions as in the preceding formulæ. Placing θ' for ψ_0 in the recurrence formula (8), the limiting angle $\frac{\psi_n}{2^n}$ is found. Then

$$\frac{\Pi_3-\mathrm{K}}{\mathrm{K}}=\frac{c}{c'\sqrt{c^2-k^2}}\left[\frac{1}{\mathrm{K}}\left(\frac{\pi}{2}-\frac{\psi_n}{2^n}\right)-(c_1'\sin\psi_1+c_2'\sin\psi_2+\ldots)\right]. \quad (17)$$

If we start with $\psi_0=\theta$ in the recurrence formula (8) then

$$\frac{\Pi_3}{\mathrm{K}}=\frac{c}{c'\sqrt{c^2-k^2}}\left[\frac{1}{\mathrm{K}}\left(\frac{\psi_n}{2^n}\right)+(c_1'\sin\psi_1+c_2'\sin\psi_2+\ldots)\right]. \quad . \quad . \quad (18)$$

It is to be noted that the angles ψ_m in (18) are not the same as the corresponding quantities in (17).

In physical problems the integral Π_3 occurs in combination with the elliptic integrals of the first and second kinds. The A.G.M. formulæ for Π_3 will, however, taken in combination with those for K and E, be useful in deriving the A.G.M. expressions in a variety of applications. Of such only the equations for the calculation of inductance will be here considered.

Application to the Calculation of Inductance.

King [4] has expressed the elliptic integral formulæ for the calculation of the self and mutual inductance of coaxial circles and coils in terms of A.G.M. series, and has

given examples of the calculations for the case of the mutual inductance of coaxial circles. The computation of the inductance of a solenoid by A.G.M. series has been treated by the writer [5]. Consequently, there will be considered here two cases only, viz., the mutual inductance of a solenoid and a coaxial circle and, second, the mutual inductance of two coaxial solenoids.

Mutual Inductance of Solenoid and Coaxial Circle.

Jones's formula [9] for the mutual inductance of a solenoid of radius A, and wound with n_1 turns per cm. on a circle of radius a, is

$$M' = 2\pi n_1 (A+a) ck \left[\frac{K-E}{k^2} + \frac{c'^2}{c^2}(K-\Pi_3)\right], \quad . \quad (19)$$

in which the modulus k of the complete elliptic integrals K, E, and Π_3 is given by

$$k^2 = \frac{4Aa}{(A+a)^2+x^2}, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

while the parameter of the integral Π_3 is

$$n = -c^2 = -\frac{4Aa}{(A+a)^2}. \quad . \quad . \quad . \quad . \quad (21)$$

Two values of M′ have to be calculated for two values of x, for which the axial distances between the plane of the circle and the ends of the solenoid are taken, and the difference gives the mutual inductance of solenoid and circle.

It is customary to calculate $(K-\Pi_3)$ by the relation

$$(K-\Pi_3) = \frac{c}{k'^2 \sin\theta' \cos\theta'}\Big[KE(k', \theta') + EF(k', \theta) - KF(k', \theta') - \frac{\pi}{2}\Big], \quad . \quad (22)$$

in which the amplitude of the incomplete integrals is $\theta' = \sin^{-1}\frac{c'}{k'}$. Introducing (22) into (19), Jones's formula takes the form given by Campbell [10]

$$M' = 2\pi n_1 (A+a)\left[\frac{c}{k}(K-E) + \frac{A-a}{x} \cdot \Psi\right], \quad . \quad (23)$$

where $$\Psi = KE(k', \theta') + EF(k', \theta') - KF(k', \theta') - \frac{\pi}{2}. \quad (24)$$

Making use of (15) Campbell's formula, expressed in terms of the A.G.M. scale $(1, k')$, reads

$$M' = \frac{\pi^2}{a_n}(A+a)n_1\left[\frac{c}{k}\left(\frac{k^2}{2}+C\right)-\frac{(A-a)}{x}\right.$$

$$\left.\{a_n(1-\sin\psi_n')-\Sigma\}\right] \quad . \quad (25)$$

in which $C = c_1{}^2+2c_2{}^2+4c_3{}^2+\ldots+2^{m-1}c_m{}^2$,

and Σ is calculated by (14), and θ' is placed for ψ_0' in the recurrence formula (11).

The corresponding formula for the A.G.M. scale $(1, k)$ is

$$M' = 2\pi n_1(A+a)K\left[\frac{c}{k}\left(1-\frac{k'^2}{2}-C'-\frac{a_n'}{K}\right)\right.$$

$$-\frac{(A-a)}{x}\left\{\frac{1}{K}\left(\frac{\pi}{2}-\frac{\psi_n}{2^n}\right)-(c_1'\sin\psi_1\right.$$

$$\left.\left.+c_2'\sin\psi_2+\ldots)\right\}. \quad (26)\right]$$

in which

$$C' = c_1'^2+2c_2'^2+4c_3'^2+\ldots+2^{m-1}c_m'^2+\ldots$$

and K is to be calculated by (4) or (7). In this formula θ' is placed for ψ_0 in the recurrence formula (8), and the limiting angle $\frac{\psi_n}{2^n}$ is found.

By the use of formulæ (16) and (18) for the integral Π_3 other formulæ for M′ are readily derived, but they offer no especial advantages.

These formulæ will be illustrated by solving for the inductance for the case[10] $A=14{\cdot}5$, $a=10$, $x=10$, and thus $\frac{A-a}{x}=0{\cdot}45$

$$k^2 = \frac{580}{24{\cdot}5^2+100}, \qquad c^2 = \frac{580}{24{\cdot}5^2},$$

$$k'^2 = \frac{120{\cdot}25}{700{\cdot}25}, \qquad c'^2 = \frac{20{\cdot}25}{600{\cdot}25},$$

$\log k = \bar{1}{\cdot}9590874$, $\log c = \bar{1}{\cdot}9925479$, $\log\sin\theta' = \bar{1}{\cdot}6466305$,

$\log k' = \bar{1}{\cdot}6174160$. $\log c' = \bar{1}{\cdot}2640464$. $\theta' = 26°\ 18'\ 36''{\cdot}83$.

Forming the A.G.M. scale $(1, k')$ as usual,

$\log\frac{b_0}{a_0}=\bar{1}\cdot6174160,$ $\quad c_1=0\cdot2928018,$ $\quad c_1^2=0\cdot08573291,$

$c_2=0\cdot0317309,$ $\quad 2c_2^2=\cdot00201371,$

$\log\frac{b_1}{a_1}=\bar{1}\cdot9591668,$ $\quad c_3=0\cdot0003728,$ $\quad 4c_3^2=\cdot00000056,$

$c_4=10^{-7},$ $\quad C=0\cdot08774718,$

$\log\frac{b_2}{a_2}=\bar{1}\cdot9995204,$ $\quad \frac{k^2}{2}=0\cdot41413785,$

$\log\frac{b_3}{a_3}=\bar{1}\cdot9999999,$ $\quad \text{Sum}=0\cdot5018850,$

$\log a_n'=\bar{1}\cdot8293644,$ $\quad \log=\bar{1}\cdot7006042,$

$\psi_n'=17°\ 35'\ 00''\cdot86,$ $\quad \log\frac{c}{k}=0\cdot0334605,$

$\log\sin\psi_n=\bar{1}\cdot4801458,$ $\quad \log\text{ first term}=\bar{1}\cdot7340647,$

$1-\sin\psi_n'=0\cdot6979034,$ $\quad \text{First term}=0\cdot5420816,$

$a_n(1-\sin\psi_n')=0\cdot4711508,$ $\quad \text{Second term}=0\cdot2026731,$

$\Sigma=0\cdot0207661,$ $\quad \text{Diff.}=0\cdot3394085.$

$\text{Diff.}=0\cdot4503847.$

$\text{Times } 0\cdot45=0\cdot2026731.$

$\Sigma=0\cdot02002255,$

$\cdot00074314,$

$\cdot00000046,$

$0\cdot02076615.$

Supposing the winding to have 10 turns per cm., the number of turns in the length x is 100, and if there be two such solenoids arranged symmetrically on either side of a concentrated winding of 1000 turns, as in the Campbell form of standard, we have to put $n_1n_2=200000$, and

$$M'=\frac{\pi^2}{a_n}(200000)(24\cdot5)(0\cdot3394085)=24\cdot31387mh.$$

A similar calculation would be necessary using the value $x=5$, and the value of M′ thus found subtracted from the above will give the mutual inductance of the above standard.

For the example just solved formula (26) is more convergent than (25). Forming the A.G.M. series $(1, k)$ with $\log k=\bar{1}\cdot9590874$, there is found

$\log\frac{b_0'}{a_0'}=\bar{1}\cdot9590874,$ $\quad c_1'=0\cdot04495175,$ $\quad C'=0\cdot00202066,$

$c_2'=0\cdot00052922.$ $\quad \cdot00000056,$

$\log\frac{b_1'}{a_1'}=\bar{1}\cdot9995184,$ $\quad =0\cdot00202122.$

$\log a_n'=\bar{1}\cdot9797845.$

The value of K calculated by formula (7) is $2\cdot3267805$,

and $\frac{a_n'}{\mathrm{K}}=0{\cdot}4102316$. Starting with $\log\sin\psi_0=\bar{1}{\cdot}6466305$, the recurrence formula (8) leads to

$$\frac{\psi_n}{2^n}=25^\circ\ 15'\ 40''{\cdot}29,$$

$$\frac{\pi}{2}-\frac{\psi_n}{2^n}=64^\circ\ 44'\ 19''{\cdot}71,$$

$$=1{\cdot}1299053 \text{ radians},$$

$$c_1'\sin\psi_1=0{\cdot}03470490,$$
$$c_2'\sin\psi_2=\ {\cdot}00051942,$$
$$\text{Sum}=0{\cdot}03522432.$$

$$1-\frac{k'^2}{2}-\mathrm{C}'-\frac{a_n'}{\mathrm{K}}=0{\cdot}5018850,$$
$$\frac{1}{\mathrm{K}}\left(\frac{\pi}{2}-\frac{\psi_n}{2^n}\right)=0{\cdot}4856088,$$
$$\text{Sum of series}=0{\cdot}0352243.$$
$$\text{Diff.}=0{\cdot}4503845.$$
$$\therefore\ \mathrm{M}'=24{\cdot}31387 \text{ m.h.}$$

$$\frac{c}{k}\left(1-\frac{k'^2}{2}-\mathrm{C}'-\frac{a_n'}{\mathrm{K}}\right)=0{\cdot}5420816,$$
$$\text{second term}=0{\cdot}2026731,$$
$$\text{Diff.}=0{\cdot}3394085.$$
$$\log=\bar{1}{\cdot}5307227,$$
$$\log 2\pi=0{\cdot}7981799,$$
$$\log(\mathrm{A}+a)=1.3891661\ ;$$
$$\log\mathrm{K}=0{\cdot}3667554,$$
$$\log n_1n_2=5{\cdot}3010300,$$
$$\log\mathrm{M}'=7{\cdot}3858541.$$

This value of M′ agrees with that found by formula (25).

The same example was worked from Campbell's formula, making use of Legendre's tables. For this purpose the moduli k and k' are placed equal to the sines of their modular angles.

Modular angle of k . . $=65^\circ{\cdot}518686$.
,, ,, k' . . $=24^\circ{\cdot}481314$.
Amplitude angle of $\theta'=26^\circ{\cdot}310231$.

The interpolated value of K, including sixth differences, was found to be 2·3267803. For E fourth differences suffice, giving the value 1·1590044. In obtaining the incomplete elliptic integrals interpolation has to made for two parameters k' and θ'. Third differences for each interpolation sufficed to give the results

$$\mathrm{F}(k',\theta')=0{\cdot}4618986$$
$$\mathrm{E}(k',\theta')=0{\cdot}4565296$$
$$\text{Diff.}=0{\cdot}0053690.$$

Making use of these values the details of the calculation are as follows :—

$$\frac{c}{k}(\mathrm{K}-\mathrm{E})=1{\cdot}2613045,$$

$$\mathrm{K}\{\mathrm{F}(k',\theta')-\mathrm{E}(k',\theta')\}=0{\cdot}0124925,$$
$$\frac{\pi}{2}=1{\cdot}5707963,$$
$$\text{Sum}=1{\cdot}5832888.$$

$$\mathrm{M}'=2\pi(24\cdot5)n_1n_2(0\cdot7897287), \qquad \mathrm{EF}(k',\theta')=0\cdot5353425,$$
$$\qquad \text{Diff.}=-\Psi=1\cdot0479463.$$
$$=24\cdot31386\,mh. \qquad \text{Times }\frac{\mathrm{A}-a}{x}=0\cdot4715758.$$

This value of M′ differs by one unit in the last place from the result found by the A.G.M. formulæ. To make the calculation using Legendre's tables required about two hours, while for that with formula (26) only forty minutes elapsed. The greater part of the time and labour using the Legendre tables has to be devoted to the interpolations, which are tedious and exacting if errors are to be avoided.

Use of a Calculating Machine.

The use of a calculating machine to form the A.G.M. scale offers no difficulty. The calculation of the limiting angle by a recurrence formula requires, however, a table of natural trigonometric functions. The writer has found the eight-place tables of Roussilhe[(6)] and Brandicourt convenient for this purpose. These are based on the decimal system (100 degrees for the quadrant), and the interval 0·01 degree is small enough to render the interpolation essentially linear.

The preceding example using Campbell's formula has been computed by both formulæ (25) and (26), and also by interpolation from Legendre's tables, making use of an eight-place calculating (adding) machine. The modular angles and the amplitude for use in Legendre's tables had to be obtained from the moduli by means of the above-mentioned tables of natural trigonometric functions in order to obtain the interpolated elliptic integrals accurate to eight places. There follow some of the steps in the calculations.

$$k=0\cdot91009649, \qquad \frac{c}{k}=1\cdot0800913,$$

$$k'=0\cdot41439640, \qquad \frac{c'}{k'}=0\cdot44323132=\sin\theta'.$$

Formula (25).	Formula (26).
$a_4=a_n=0\cdot67509433,$	$a_3'=a_n'=0\cdot95451894,$
$\mathrm{C}=0\cdot087747159,$	$\mathrm{C}'=0\cdot002021221,$
$\sin\psi_n'=0\cdot30209663,$	$\frac{\pi}{2}-\frac{\psi_n}{2^n}=1\cdot1299053,$
$\Sigma=0\cdot020766009,$	$\Sigma(c'_m\sin\psi_m)=0\cdot035224255,$
$\mathrm{K}=2\cdot3267805,$	$\mathrm{K}=2\cdot3267805,$
$\mathrm{M}'=24\cdot313871,$	$\mathrm{M}'=24\cdot313869.$

By Legendre's Tables.

$\angle k=65°\cdot5186894$,	$K=2\cdot3267805$,
$\angle k'=24°\cdot4813108$,	$E=1\cdot1590043$,
$\theta'=26°\cdot3102348$,	$F(k', \theta')=0\cdot46189868$,
$\Psi=1\cdot0479463$,	$E(k', \theta')=0\cdot45652967$.
$M'=24\cdot313869$.	

The calculations were made complete for each method, thus checking parts of the calculations common to more than one method. The required times for the computations were for formulæ (25), (26), and the Legendre tables, $2\frac{1}{4}$ hours, 3 hours, and $3\frac{1}{2}$ hours, respectively. It is probable that all these times would be much smaller with a more modern and elaborate machine.

Mutual Inductance of Coaxial Solenoids.

If the radii of the solenoids are A, a, their axial lengths P, S, their winding densities n_1, n_2, and the distance between their centres x_0, their mutual inductance may be conveniently written in the form [11]

$$M=2\pi^2a^2n_1n_2[r_1B_1-r_2B_2-r_3B_3+r_3B_4] \quad . \quad (27)$$

in which

$$r_n=\sqrt{x_n^2+A^2}, \quad x_1=x_0-\left(\frac{S+P}{2}\right), \quad x_3=x_0+\left(\frac{S-P}{2}\right),$$

$$x_2=x_0-\left(\frac{S-P}{2}\right), \quad x_4=x_0+\left(\frac{S+P}{2}\right),$$

and B_n is a function of the parameters $\alpha=\frac{a}{A}$ and $\xi_n=\frac{x_n}{A}$, which involves elliptic integrals of all three kinds. Using King's formulæ, two A.G.M. series expressions for B_n may be found.

Using the previous nomenclature the moduli of the integrals are

$$k^2=\frac{4\alpha}{(1+\alpha)^2+\xi_n^2}, \quad k'^2=\frac{(1-\alpha)^2+\xi_n^2}{(1+\alpha)^2+\xi_n^2}, \quad c'^2=\frac{(1-\alpha)^2}{(1+\alpha)^2},$$

$$c^2=\frac{4\alpha}{(1+\alpha)^2}, \qquad \sin\theta'=\frac{c'}{k'}.$$

If the A.G.M. scale $(1, k')$ be used.

$$B_n=\frac{1}{3\alpha a_n}\sqrt{\frac{(1+\alpha)^2+\xi_n^2}{1+\xi_n^2}}\left[\frac{4\alpha+3\xi_n^2}{(1+\alpha)^2+\xi_n^2}+\frac{C}{2\alpha}\{\xi_n^2-2\right.$$

$$\left.\times(1+\alpha^2)\}-\frac{3}{2}\frac{\xi_n(1-\alpha^2)}{\alpha\sqrt{(1+\alpha)^2+\xi_n^2}}\{a_n(1-\sin\psi_n')-\Sigma\}\right], \quad (28)$$

where C is given by (25 a) and Σ by (14), the recurrence formula (11) is used, starting with θ' for ψ_0'.

If the calculation is based on the scale (1, k),

$$\begin{aligned} B_n = & \frac{2(1+\alpha^2)-\xi_n^2}{3\pi\alpha^2}\sqrt{\frac{(1+\alpha)^2+\xi_n^2}{1+\xi_n^2}} \cdot a_n'\left[1+\frac{K}{a_n'}\left\{\left(\frac{k'^2}{2}+C\right)\right.\right. \\ & \left.+\frac{\xi_n^2-2(1-\alpha)^2}{2(1+\alpha^2)-\xi_n^2}\right\} - \frac{3\xi_n(1-\alpha^2)}{a_n'[2(1+\alpha^2)-\xi_n^2]\sqrt{(1+\alpha)^2+\xi_n}} \\ & \times\left.\left\{\left(\frac{\pi}{2}-\frac{\psi_n}{2^n}\right)-K(c_1'\sin\psi_1+c_2'\sin\psi_2+\ldots)\right\}\right]. \qquad (29) \end{aligned}$$

in which the recurrence formula (8) is employed, starting with $\theta'=\psi_0$, while C$'$ is obtained from (26 a).

Formulæ (28) and (29) agree, and give results of great accuracy. It is evident, however, that the calculations by these formulæ resemble those that have gone before, and offer no new points of interest. Enough has been given to show that, by the use of A.G.M. series formulæ, calculations of inductance may be made with a precision limited only by that of the means employed in the computations. If logarithmic tables are used, no further tables are necessary; if a calculating machine is used, tables of natural trigonometric functions are also necessary, but these are also required with a machine calculation made in connexion with Legendre's tables. The advantage in saving of time seems to lie distinctly with the A.G.M. formulæ.

References.

(1) The tables of the elliptic integrals K and E by H. Nagaoka and S. Sakurai, Institute of Physical and Chemical Research, Table I., Tokyo, 1922, are given to seven places and have the especial convenience that they make use of k^2 as argument, instead of the modular angle. The Tables XII. and XIII. in the Bureau of Standards Sci. Paper 169 are taken from Legendre.

(2) "A. M. Legendres Tafeln der Elliptischen Normalintegrale erster und zweiter Gattung," edited by Fritz Emde, Konrad Wittwer, Stuttgart (1931).

(3) Gauss, 'Gesammelte Werke,' Bd. iii. pp. 357–360 (1818).

(4) L. V. King, "On the Direct Numerical Calculation of Elliptic Functions and Integrals," Cambridge University Press (1924); "Calculation of the Mutual Inductance of Coaxial Circles in A.G.M. Series," Proc. Roy. Soc. A, vol. c. p. 63 (1921); "On some new Formulæ for the Calculation of Self and Mutual Induction of Coaxial Circular Coils," Phil. Mag. vol. xv. p. 1097 (1933).

(5) Julius Bauschinger and J. Peters. Leipzig, W. Engelmann; New York, G. E. Stechert, 1910.

(6) "Tables à 8 décimales des valeurs naturelles des sinus, cosinus et tangentes dans le systeme décimal, de centigrade en centigrade de 0 à 100 grades," by Roussilhe and Brandicourt. Section de Géodésie de l'Union Géodésique et Géophysique Internationale. Dorel, Paris (1925).

(7) That of the mutual inductance of an Ayrton-Jones current balance, Bureau of Standards Sci. Paper 169, p. 106 (1912). The value of $E(k', \theta)$ there obtained by interpolation from Legendre's tables is in error by a unit in the last place.

(8) "Methods for the Derivation and Expansion of Formulas for the Mutual Inductance of Coaxial Circles and for the Inductance of Single-layer Solenoids," Bureau of Standards Jour. of Research, vol. i. no. 16, p. 505 (Oct. 1928).

(9) J. V. Jones, Proc. Roy. Soc. vol. lxiii. p. 198 (1898); also Trans. Roy.Soc. 182, A (1891). See also Bur. Standards Sci. Paper 169, pp. 99 and 106.

(10) A. Campbell, Proc. Roy. Soc. A, vol. lxxix. p. 428 (1907), describes a new form of mutual inductance standard. The details of the calculation of the inductance of this are given on pp. 1128–1130. The same problem is solved by the use of Legendre's tables in the Bur. Standards Sci. Paper 169, p. 109. The interpolated values of F and E there given are slightly in error, giving a value of M, which is too large (see values given at the bottom of the page).

(11) The absolute formula for the mutual inductance of coaxial solenoids has been given in various forms, see H. Nagaoka, Jour. Coll. Sci. Tokyo, vol. xxvii. art 6 (1909); G. R. Olshausen, Phys. Rev. vol. xxxi. pp. 617–636 (1910), vol. xxxv. p. 148 (1912). The form here given follows J. E. Clem, "Mechanical Forces in Transformers," Jour. A. I. E. E. vol. xlvi. p. 814 (Aug. 1927), and has the advantage that the quantity B, which is of zero dimensions, does not depart very much from unity, and is well adapted for purposes of tabulation.

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XCI. *The Conservation of Probability and Energy as Criteria for the Validity of Wave Equations.* By A. Lees, *B.Sc., Research Scholar, University College, Nottingham* †.

1. *Two Conditions a Wave Equation should satisfy.*

In quantum mechanics, the principle of conservation of probability (or of the "statistical charge-density" in the case of one electron) leads to the condition that the probability

$$\rho = \psi\psi^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

must satisfy an equation of the form

$$\operatorname{div} \sigma + \frac{\partial \rho}{\partial t} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

† Communicated by Prof. H. T. H. Piaggio, M.A., D.Sc.

Since this equation must be true in virtue of the wave equation for the system, a condition is imposed upon the latter equation.

Eddington † regards a wave equation as expressing the conservation of probability, but there is also a second condition which the wave equation must satisfy, and which, in those cases in which the Hamiltonian does not involve the time explicitly, expresses the conservation, not of probability, but of energy. We derive this condition as follows.

Dirac ‡ shows that, if the representation be suitably chosen, any observable ξ, which does not involve the time explicitly, satisfies the equation

$$(ih/2\pi)\dot{\xi} = \xi \mathrm{H} - \mathrm{H}\xi, \quad \ldots \ldots \quad (3)$$

H being the Hamiltonian, which is also an observable; and that any wave function ψ satisfies the wave equation

$$\frac{ih}{2\pi}\frac{\partial \psi^*}{\partial t} = \mathrm{H}\left(q, -\frac{ih}{2\pi}\frac{\partial}{\partial q}, t\right)\psi^*, \quad \ldots \quad (4)$$

where $\mathrm{H}(q, p, t)$ is the Hamiltonian. [ψ^* is equivalent to Dirac's $(q'|)$.]

The generalization of (3) for an observable ξ which may involve t explicitly is evidently

$$\left(\frac{ih}{2\pi}\right)\left(\dot{\xi} - \frac{\partial \xi}{\partial t}\right) = \xi \mathrm{H} - \mathrm{H}\xi. \quad \ldots \ldots \quad (3\,a)$$

Substituting H for ξ in (3 a) gives

$$\dot{\mathrm{H}} = \frac{\partial \mathrm{H}}{\partial t}. \quad \ldots \ldots \ldots \quad (5)$$

Therefore, if Φ and Ψ be the state symbols corresponding to the state whose wave function is ψ, (5) gives

$$\frac{d}{dt}(\Phi \mathrm{H} \Psi) = \Phi \dot{\mathrm{H}} \Psi = \Phi \frac{\partial \mathrm{H}}{\partial t} \Psi. \quad \ldots \ldots \quad (6)$$

[If H does not involve t explicitly, $\frac{\partial \mathrm{H}}{\partial t} = 0$, and (6) leads to the equation

$$\Phi \mathrm{H} \Psi = \text{a constant}. \quad \ldots \ldots \quad (6\,a)$$

Since $\Phi \mathrm{H} \Psi$ is the average value of H for determinations of the value at time t in the given state §, (6 a) expresses the

† Proc. Roy. Soc. A, cxxxviii. p. 41 (1932).
‡ Dirac, 'Quantum Mechanics,' pp. 97 and 112.
§ Dirac, 'Quantum Mechanics,' p. 32

constancy of this average value. This is the quantum-mechanical analogue of the classical conservation of energy.]

Also, in Dirac's notation,

$$\Phi \mathrm{H} \Psi = \int (|q')\, dq' \cdot \frac{ih}{2\pi} \frac{\partial}{\partial t} (q'|) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

and

$$\Phi \frac{\partial \mathrm{H}}{\partial t} \Psi = \int (|q')\, dq' \left\{ \frac{\partial}{\partial t} \mathrm{H}\left(q', -\frac{ih}{2\pi} \frac{\partial}{\partial q'}, t\right) \right\} (q'|). \quad (7a)$$

Again, the $(|q')$ and $(q'|)$ of Dirac's notation are equivalent to the usual ψ and ψ^*; therefore, from (6), (7), and (7*a*), changing † dq' into $d\tau$,

$$\frac{d}{dt} \int \psi \frac{\partial \psi^*}{\partial t} d\tau = \frac{2\pi}{ih} \int \psi \left\{ \frac{\partial}{\partial t} \mathrm{H}\left(q, -\frac{ih}{2\pi} \frac{\partial}{\partial q}, t\right) \right\} \psi^* d\tau. \quad . \; . \; . \quad (8)$$

If H does not involve t explicitly, (8) reduces to

$$\frac{d}{dt} \int \psi \frac{\partial \psi^*}{\partial t} d\tau = 0. \quad . \; . \; . \; . \; . \; . \quad (8\,a)$$

Hence our wave equation (4) must be such that, for any wave function ψ satisfying it, conditions (2) and (8) are satisfied.

2. *The First Condition for the Non-relativistic Wave Equation.*

It is well known that in the case of Schrödinger's non-relativistic wave equation (in the form sometimes called the *time equation*) for a single electron,

$$\frac{\partial \psi}{\partial t} = \mathrm{L}(\psi), \qquad \mathrm{L} \equiv \frac{h}{4\pi i m}\left(\Delta - \frac{8\pi^2 m}{h^2} \mathrm{V}\right), \quad . \; . \quad (9)$$

the condition (2) is satisfied. We have, in fact ‡,

$$\operatorname{div} \mathrm{S} + \frac{\partial}{\partial t}(\psi\psi^*) = 0, \quad \mathrm{S} = \frac{h}{4\pi i m}(\psi \operatorname{grad} \psi^* - \psi^* \operatorname{grad} \psi). \quad . \; . \; . \quad (10)$$

† This is permissible, because in Dirac's work the q's are Cartesian coordinates, and so the ratio of dq' to $d\tau$ is constant.

‡ Sommerfeld, 'Wave Mechanics,' p. 89, eqn. (7).

3. *The First Condition for the Second-order Relativistic Equation.*

In the case of the second-order relativistic equation †

$$\sum_1^4 \left(\frac{\partial}{\partial x_k} + i\phi_k\right)^2 \psi - A^2\psi = 0, \quad . \quad . \quad . \quad (11)$$

where $\quad x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict,$

$$\phi_r = \frac{2\pi e}{hc} A_r, \quad (r = 1, 2, 3), \quad \phi_4 = \frac{2\pi i e}{hc}\phi, \quad A = \frac{2\pi m_0 c}{h},$$

and A_1, A_2, A_3, ϕ are the vector and scalar potentials, the result corresponding to (10) is

$$\operatorname{div} S = 0, \quad S_k = \frac{h}{4\pi i m_0}\left\{\psi \frac{\partial\psi^*}{\partial x_k} - \psi^* \frac{\partial\psi}{\partial x_k} - 2i\phi_k\psi\psi^*\right\}, \quad . \quad . \quad . \quad (12)$$

where $\operatorname{div} S$ is a four-dimensional divergence.

The x_4 component does *not* reduce to a constant multiple of $\psi\psi^*$. In fact, it has no simple interpretation. So, in this case, condition (2) is not satisfied.

4. *The First Condition for Dirac's Equation.*

In view of the many notations employed in connexion with Dirac's linear wave equation ‡, we shall first specify our notation completely. We write the equation as

$$\left(\sum_1^4 \alpha_r \Omega_r + A\right)\psi = 0, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$\Omega_r \equiv \frac{\partial}{\partial x_r} + i\phi_r,$$

and the α's are fourth-order matrices satisfying the equations

$$\alpha_r^2 = 1, \ (r = 1, 2, 3, 4); \quad \alpha_r\alpha_s = -\alpha_s\alpha_r, \ (r \neq s). \quad (14)$$

We further assume that the α's are "four-point" matrices, and such that α_1, α_2, and α_3 are symmetrical, having real elements, while α_4 is antisymmetrical with imaginary elements §.

† *Ibid.* pp. 100–102.

‡ Sommerfeld, 'Wave Mechanics,' pp. 257–259.

§ Our notation is that employed by Sommerfeld, *ibid.* p. 258, eqn. (5), but we use different a's. For a justification of our assumptions regarding the a's, *cf.* Eddington, Proc. Roy. Soc. A, cxxxiii. p. 311 (1931). We may define our a's in terms of his E's by the equation

$$a_\mu = iE_\mu, \quad (\mu = 1, 2, 3, 4).$$

Condition (2) is then satisfied. We have, in fact,

$$\operatorname{div}(\psi^* \alpha_4 \alpha_\mu \psi) = 0, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and the x_4 component is

$$\psi^* \alpha_4 \alpha_4 \psi = \psi^* \psi = \psi \psi^*,$$

since $\alpha_4^2 = 1$.

Equation (15) is not easily transcribed from one notation to another, so we give a proof in our present notation.

Let θ_r be defined by the equations

$$\theta_r = 1, \ (r = 1, 2, 3), \quad \theta_4 = -1. \quad . \quad . \quad . \quad (16)$$

From our definitions and assumptions, it then follows that

$$\phi_r^* = \theta_r \phi_r, \quad \left(\frac{\partial}{\partial x_r}\right)^* \equiv \theta_r \frac{\partial}{\partial x_r}, \quad \alpha_r^* = \theta_r \alpha_r, \ (r = 1, 2, 3, 4). \quad . \quad . \quad . \quad (17)$$

Equation (13) gives

$$\left\{ \sum_1^4 \alpha_r \left(\frac{\partial}{\partial x_r} + i\phi_r \right) + \mathrm{A} \right\} \psi = 0. \quad . \quad . \quad . \quad (13\,a)$$

If we now write down the equation conjugate to (13 a), using (17), we obtain, since $\theta_r^2 = 1$, ($r = 1, 2, 3, 4$),

$$\left\{ \sum_1^4 \alpha_r \left(\frac{\partial}{\partial x_r} - i\phi_r \right) + \mathrm{A} \right\} \psi^* = 0. \quad . \quad . \quad (13\,b)$$

We now multiply (13a) by $\psi^* \alpha_4$ on the left, and (13b) by $\alpha_4 \psi$ on the right, obtaining respectively

$$\psi^* \alpha_4 \left\{ \sum_1^4 \alpha_r \left(\frac{\partial}{\partial x_r} + i\phi_r \right) + \mathrm{A} \right\} \psi = 0, \quad . \quad (18)$$

$$\left[\left\{ \sum_1^4 \alpha_r \left(\frac{\partial}{\partial x_r} - i\phi_r \right) + \mathrm{A} \right\} \psi^* \right] (\alpha_4 \psi) = 0. \quad (18\,a)$$

Now, in view of the properties assigned to our α's, we have

$$(\alpha_r)_{\mu\nu} = \theta_r (\alpha_r)_{\nu\mu}, \quad (r = 1, 2, 3, 4). \quad . \quad . \quad (19)$$

$$\therefore \quad (\psi^* \alpha_r)_\mu = \sum_{\nu=1}^4 \psi_\nu^* (\alpha_r)_{\nu\mu} = \theta_r \sum_{\nu=1}^4 \psi_\nu^* (\alpha_r)_{\mu\nu} = \theta_r (\alpha_r \psi^*)_\mu.$$

$$\therefore \quad \psi^* \alpha_r = \theta_r \alpha_r \psi^*, \quad (r = 1, 2, 3, 4). \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also, evidently

$$\alpha_r \alpha_4 = -\theta_r \alpha_4 \alpha_r, \quad (r = 1, 2, 3, 4). \quad . \quad . \quad (21)$$

From (20) and (21), we have, since $\theta_r^2 = 1$,

$$(\alpha_r \psi^*) \alpha_4 = \theta_r \psi^* \alpha_r \alpha_4 = -\psi^* \alpha_4 \alpha_r. \quad . \quad . \quad . \quad (22)$$

Applying (22) to (18 a), we have

$$\left[-\sum_1^4\left(\frac{\partial}{\partial x_r}-i\phi_r\right)\psi^*\alpha_4\alpha_r+\mathrm{A}\psi^*\alpha_4\right]\psi=0. \quad . \ (18\,b)$$

Subtracting the left side of (18 b) from that of (18), we have

$$\operatorname{div}(\psi^*\alpha_4\alpha_\mu\psi)=0,$$

as required.

5. *The Second Condition for the Non-relativistic Equation.*

Schrödinger's equation (9) applies only to the case in which there are no magnetic forces, *i. e.*,

$$\phi_1=\phi_2=\phi_3=0.$$

In view of the condition

$$\sum_1^4\frac{\partial\phi_\mu}{\partial x_\mu}=0, \quad . \ . \ . \ . \ . \ . \ . \ (23)$$

it follows that

$$\frac{\partial\phi_4}{\partial x_4}=0,$$

i. e.,

$$\frac{\partial \mathrm{V}}{\partial t}=e\frac{\partial\phi}{\partial t}=0. \quad . \ . \ . \ . \ . \ . \ . \ (24)$$

We have, from (9),

$$\frac{\partial\psi}{\partial t}=\mathrm{L}(\psi)\,;$$

so, writing down the conjugate equation,

$$\frac{\partial\psi^*}{\partial t}=\mathrm{L}^*(\psi^*)=-\mathrm{L}(\psi^*). \quad . \ . \ . \ . \ (9\,a)$$

Comparison of (9 a) with (4) gives

$$\mathrm{H}\left(q,-\frac{ih}{2\pi}\frac{\partial}{\partial q},\ t\right)\equiv-\frac{ih}{2\pi}\mathrm{L}. \quad . \ . \ . \ . \ . \ . \ (25)$$

Hence, as $\dfrac{\partial \mathrm{V}}{\partial t}=0$, H does not involve t explicitly, and so the condition to be satisfied is given by (8 a).

If $\mathrm{L}(\psi^*)=v$, we have, from (9) and (9 a),

$$\begin{aligned}\frac{\partial}{\partial t}\left(\psi\frac{\partial\psi^*}{\partial t}\right)&=\psi\mathrm{L}\{\mathrm{L}(\psi^*)\}-\mathrm{L}(\psi)\mathrm{L}(\psi^*)\\&=\psi\mathrm{L}(v)-v\mathrm{L}(\psi). \quad . \ . \ . \ . \ . \ . \ (26)\end{aligned}$$

Since L is self-adjoint,

$$\psi \mathrm{L}(v) - v\mathrm{L}(\psi) = \operatorname{div} \mathrm{U}, \quad . \quad . \quad . \quad . \quad (27)$$

where U is a certain space vector †; hence, from (26) and (27), we have, on integrating throughout infinite space,

$$\frac{d}{dt}\int \psi \frac{\partial \psi^*}{\partial t} d\tau = \int \operatorname{div} \mathrm{U}\, d\tau = \int \mathrm{U}_n dS, \quad . \quad . \quad (28)$$

the latter integral being over the "surface at infinity." But U will have similar boundary properties to those of ψ and ψ^*, and therefore

$$\int \mathrm{U}_n dS = 0.$$

Hence (28) gives

$$\frac{d}{dt}\int \psi \frac{\partial \psi^*}{\partial t} d\tau = 0, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

i. e., condition (8 *a*) is satisfied.

6. *The Second Condition for the Second-order Relativistic Equation.*

In the case of the second-order relativistic wave equation (11), the condition (8), like condition (2), is not satisfied. We cannot attempt to establish any relation by the method employed in § 5, since the wave equation is no longer of the form

$$\frac{\partial \psi}{\partial t} = \mathrm{L}(\psi).$$

7. *The Second Condition for Dirac's Equation.*

We now show that, in the case of Dirac's linear wave equation, condition (8) is satisfied. We form, from equation (13), a second-order wave equation in the well-known way ‡ by applying on the left the operator

$$\sum_1^4 \alpha_r \Omega_r - \mathrm{A}.$$

We obtain

$$\left(\sum_1^4 \alpha_r \Omega_r - \mathrm{A}\right)\left(\sum_1^4 \alpha_r \Omega_r + \mathrm{A}\right)\psi = 0,$$

or §

$$\left\{\sum_1^4 \Omega_r^2 - \mathrm{A}^2 + i \sum_{r,s} \alpha_r \alpha_s \left(\frac{\partial \phi_s}{\partial x_r} - \frac{\partial \phi_r}{\partial x_s}\right)\right\}\psi = 0. \quad . \quad (30)$$

† Sommerfeld, 'Wave Mechanics,' p. 105, eqn. (16).

‡ Sommerfeld, 'Wave Mechanics,' p. 258.

§ *Ibid.* p. 260.

We also write down the equation conjugate to (30). This is, using (17),

$$\left\{\sum_1^4 \Omega_r^{*2} - A^2 - i\sum_{r,s} \alpha_r\alpha_s\left(\frac{\partial\phi_s}{\partial x_r} - \frac{\partial\phi_r}{\partial x_s}\right)\right\}\psi^* = 0. \quad (30\,a)$$

We now multiply the left side of (30) by ψ^* on the left, that of (30 a) by ψ on the left, and subtract. We obtain

$$\mathrm{I} + i\mathrm{II} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where †

$$\mathrm{I} = \operatorname{div}\left(\psi^*\frac{\partial\psi}{\partial x_k} - \psi\frac{\partial\psi^*}{\partial x_k} + 2i\phi_k\psi\psi^*\right), \quad . \quad . \quad (31\,a)$$

$$\mathrm{II} = \sum_{r,s}\left(\frac{\partial\phi_s}{\partial x_r} - \frac{\partial\phi_r}{\partial x_s}\right)(\psi^*\alpha_r\alpha_s\psi + \psi\alpha_r\alpha_s\psi^*). \quad . \quad (31\,b)$$

Now we have, from (19),

$$\psi\alpha_r\alpha_s\psi^* = \sum_{\mu=1}^4\sum_{\nu=1}^4\sum_{\sigma=1}^4 \psi_\mu(\alpha_r)_{\mu\nu}(\alpha_s)_{\nu\sigma}\psi_\sigma^*$$

$$= \theta_r\theta_s\sum_{\mu=1}^4\sum_{\nu=1}^4\sum_{\sigma=1}^4 \psi_\mu(\alpha_r)_{\nu\mu}(\alpha_s)_{\sigma\nu}\psi_\sigma^*$$

$$= \theta_r\theta_s\psi^*\alpha_s\alpha_r\psi.$$

Also, for all non-zero terms of II, $r \neq s$, and so $\alpha_s\alpha_r = -\alpha_r\alpha_s$.

$$\therefore \qquad \mathrm{II} = \sum_{r,s}\left(\frac{\partial\phi_s}{\partial x_r} - \frac{\partial\phi_r}{\partial x_s}\right)\psi^*\alpha_r\alpha_s\psi(1-\theta_r\theta_s). \quad . \quad . \quad (31c)$$

$1-\theta_r\theta_s$ vanishes unless r, s are one of the pairs of numbers (4, 1), (4, 2), (4, 3), and for these values it has the value 2.

Hence, putting $r=4$, and changing s to k, (31c) gives

$$\mathrm{II} = 2\sum_{k=1}^4\left(\frac{\partial\phi_k}{\partial x_4} - \frac{\partial\phi_4}{\partial x_k}\right)\psi^*\alpha_4\alpha_k\psi,$$

since the term $k=4$ vanishes ; so, using (15),

$$\mathrm{II} = 2\sum_{k=1}^4\frac{\partial\phi_k}{\partial x_4}\psi^*\alpha_4\alpha_k\psi - 2\sum_{k=1}^4\frac{\partial}{\partial x_k}(\phi_4\psi^*\alpha_4\alpha_k\psi). \quad (31\,d)$$

From (31), (31 a), and (31 d), we have ‡,

$$\operatorname{div}\left(\psi^*\frac{\partial\psi}{\partial x_k} - \frac{\psi\partial\psi^*}{\partial x_k} + 2i\phi_k\psi\psi^* - 2i\phi_4\psi^*\alpha_4\alpha_k\psi\right)$$

$$= -2i\sum_{k=1}^4\frac{\partial\phi_k}{\partial x_4}\psi^*\alpha_4\alpha_k\psi. \quad . \quad . \quad . \quad . \quad (32)$$

† *Ibid.* p. 106, eqn. (22 a).

‡ This is not the result obtained by Landé, *Zs. f. Phys.* xlviii. p. 601, eqn. (12) (1928). The difference seems to be due to an error in his eqn. (11 a), in which the second M should be $\bar{\mathrm{M}}$.

The x_4 component of the four-vector on the left side of (32) reduces, on putting $\alpha_4^2=1$, to

$$\psi^*\frac{\partial\psi}{\partial x_4}-\psi\frac{\partial\psi^*}{\partial x_4};$$

so we obtain, on integrating both sides of (32) throughout infinite space, and applying Green's theorem and the boundary conditions satisfied by ψ and ψ^*,

$$\frac{d}{dx_4}\int\left(\psi^*\frac{\partial\psi}{\partial x_4}-\psi\frac{\partial\psi^*}{\partial x_4}\right)d\tau=-2i\int\sum_{k=1}^{4}\frac{\partial\phi_k}{\partial x_4}\psi^*\alpha_4\alpha_k\psi\,d\tau. \quad . \quad . \quad . \quad (33)$$

Again, we have

$$\int\left(\psi^*\frac{\partial\psi}{\partial x_4}-\psi\frac{\partial\psi^*}{\partial x_4}\right)d\tau=-\frac{d}{dx_4}\int\psi\psi^*d\tau+2\int\psi^*\frac{\partial\psi}{\partial x_4}d\tau, \quad . \quad . \quad . \quad (34)$$

and in view of the conservation of $\psi\psi^*$ the first term on the right vanishes, so, from (33) and (34), we have

$$\frac{d}{dx_4}\int\psi^*\frac{\partial\psi}{\partial x_4}d\tau=-i\int\sum_{k=1}^{4}\frac{\partial\phi_k}{\partial x_4}\psi^*\alpha_4\alpha_k\psi\,d\tau. \quad . \quad (35)$$

We shall require the equation conjugate to (35), which is, using (17),

$$\frac{d}{dx_4}\int\psi\frac{\partial\psi^*}{\partial x_4}d\tau=i\int\sum_{k=1}^{4}\frac{\partial\phi_k}{\partial x_4}\psi\alpha_4\alpha_k\psi^*\,d\tau. \quad (35\,a)$$

Our wave equation (13) may be written

$$\left\{\sum_1^3\alpha_r\Omega_r+\alpha_4\left(\frac{\partial}{\partial x_4}+i\phi_4\right)+\mathrm{A}\right\}\psi=0, \quad . \quad . \quad (13\,c)$$

or, multiplying on the left by α_4, putting $\alpha_4^2=1$,

$$\frac{\partial\psi}{\partial x_4}=\left(-\sum_1^3\alpha_4\alpha_r\Omega_r-i\phi_4-\mathrm{A}\alpha_4\right)\psi. \quad . \quad . \quad (13\,d)$$

The equation conjugate to (13 *d*) is, using (17),

$$\frac{\partial\psi^*}{\partial x_4}=\left(-\sum_1^3\alpha_4\alpha_r\Omega_r^*+i\phi_4-\mathrm{A}\alpha_4\right)\psi^*. \quad . \quad . \quad (13\,e)$$

Comparison of (13 *e*) with (4) gives

$$\mathrm{H}\left(q,\,-\frac{ih}{2\pi}\frac{\partial}{\partial q},\,t\right)\equiv\frac{hc}{2\pi}\left(\sum_1^3\alpha_4\alpha_r\Omega_r^*-i\phi_4+\mathrm{A}\alpha_4\right). \quad (36)$$

Thus H involves t explicitly only in the components of the four-potential, and

$$\frac{\partial}{\partial t}\mathrm{H}\left(q, -\frac{ih}{2\pi}\frac{\partial}{\partial q}, t\right) \equiv -\frac{hc}{2\pi}\left(\sum_1^3 \alpha_4\alpha_r i\frac{\partial\phi_r}{\partial t} + i\frac{\partial\phi_4}{\partial t}\right),$$

i. e.,

$$\frac{\partial}{\partial t}\mathrm{H}\left(q, -\frac{ih}{2\pi}\frac{\partial}{\partial q}, t\right) \equiv -\frac{hc}{2\pi}\cdot i\alpha_4\sum_1^4 \alpha_k\frac{\partial\phi_k}{\partial t}, \quad . \quad . \quad . \quad (37)$$

since $\alpha_4^2=1$.

From (35 a) and (37), we have

$$\frac{d}{dx_4}\int\psi\frac{\partial\psi^*}{\partial x_4}d\tau = \int\psi\left\{\frac{\partial}{\partial t}\mathrm{H}\left(q, -\frac{ih}{2\pi}\frac{\partial}{\partial q}, t\right)\right\}\psi^* d\tau\,.\,\frac{2\pi i}{hc^2},$$

i. e.,

$$\frac{d}{dt}\int\psi\frac{\partial\psi^*}{\partial t}d\tau = \frac{2\pi}{ih}\int\psi\left\{\frac{\partial}{\partial t}\mathrm{H}\left(q, -\frac{ih}{2\pi}\frac{\partial}{\partial q}, t\right)\right\}\psi^* d\tau. \quad . \quad . \quad . \quad (38)$$

This is precisely our condition (8), which therefore, like condition (2), is satisfied in the case of Dirac's equation.

8. *Summary.*

There are two conditions which a wave equation should satisfy, one corresponding to the conservation of probability, the other expressing a relation which, in the case in which the Hamiltonian does not involve the time explicitly, corresponds to the conservation of energy. It is known that the first condition is satisfied by Schrödinger's non-relativistic equation and by Dirac's equation, but not by the second-order relativistic equation. We prove that the second condition also is satisfied in the same two cases. These facts seem to provide a stronger reason for rejecting the second-order relativistic equation than those usually given.

One of our results disagrees with Landé's work. We point out the source of his error.

In conclusion, I wish to express my thanks to Professor H. T. H. Piaggio, M.A., D.Sc., for much valuable criticism in connexion with this paper.

University College, Nottingham.
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XCII. *A Unitary Principle linking Relativity, Gravitation, and the Discreteness of Quanta.* By A. Press *.

Summary.

It is shown that the Kaufmann-Bucherer experiments really prove that in addition to a resultant modification of the electrical force effect at a distance, due to a moving charge q, satisfying the classical equation (Heavisidean),

$$q=q_0\left(1-\frac{1}{3}\frac{v^2}{c^2}\right),$$

that there is similarly a resultant modification of the gravitational force at a distance which can be expressed by a (so-called Einsteinian) relation

$$m_e=m\left(1-\frac{1}{4}\frac{v^2}{c^2}\right)\equiv m(1-\sqrt{1-\beta^2}).$$

Both effects substantiate the author's theory that at right angles to the usual central force system P, gravitationally and electrically, there exists a Q force which depends in magnitude on the time rate of change of the radius vector. Thus no need is found for deviating from Newton's conceptions of time, space, and mass, either astronomically or electronically. In fact, with the author's gravitational equation, better agreement is achieved astronomically than with the Einsteinian.

The new theory is not only fruitful in furthering the correlation between Einstein's ideas and classical theory, but it actually leads, as a matter of course, to Bohr's conception of discrete orbital states. Moreover, for the order of atomic distances discussed by Sommerfeld and Kratzer, the force function naturally reduces to the theoretically derived form

$$F=\frac{dV}{dr}=P\left(1-\frac{C_3}{r^3}+\frac{C_5}{r^5}\right).$$

The second and fourth inverse powers do not occur at all. This is in keeping with the experimental deductions of Born and Heisenberg.

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1. IN a central force system of Newtonian type, we have a force P_g' defined by the relation gravitationally

$$P_g'=Gm'.mu^2, \quad \ldots\ldots \quad (1.1)$$

$$u=1/r. \quad \ldots\ldots\ldots \quad (1.2)$$

* Communicated by the Author.

For the electrical case correspondingly, we have with a force P_e',

$$P_e' = q'/k \cdot qu^2. \quad \ldots \ldots \quad (1.3)$$

The Kaufmann-Bucherer experiments (see Phil. Mag., Nov. 1932, p. 758) have shown that the simply stated laws (1.1) and (1.3) need some type of supplement rather than outright modification. That is to say, it will be shown the Newtonian concepts of time and space do not have to be abandoned at all—that, in fact, the laws (1.1) and (1.3) still can hold true for the extremely high velocities approaching the velocity of light. However, the thing that does have to be postulated in the "Dynamics of a Particle" (see treatise by Tait and Steele, p. 118) is that, in addition to such radial forces P giving the usual Newtonian effects, a supplemental force Q exists at right angles to the radius vector and dependent in magnitude on $\dfrac{dr}{dt}$ (see Phil. Mag., Jan. 1931). In this way only can the orbital equation of Einstein be derived *without further hypothesis*. In fact, besides the mathematical necessity for discrete orbital states (Bohr's hypothesis), the existence is proved of a repulsional force factor varying with the inverse third power of the radius vector together with an inverse fifth power which hitherto were simply assumed when dealing with intra-atomic forces (Sommerfeld) or inter-atomic forces (Kratzer).

2. Consider, then, in the first instance the classical evidences of the Kaufmann-Bucherer experiments. Following the ideas of a Heaviside and a Maxwell, it was shown theoretically that, due to a translatory velocity v of sufficient magnitude, the electrical charge q of (1.3) became effectively modified, so far as distant force resultants were concerned, that it satisfied the deduced formula

$$q = q_0\left(1 - \frac{1}{3}\frac{v^2}{c^2}\right), \quad \ldots \ldots \quad (2.1)$$

where c is the velocity of light. Quite empirically, however, it was then indicated that the experiments demanded a working formula rather different from (2.1). In fact, rather remarkable concordance obtained if instead of the factor 1/3 in (2.1) the factor ·58 was employed. Thus the formula needed to read

$$q = q_0\left(1 - \cdot 58\,\frac{v^2}{c^2}\right). \quad \ldots \ldots \quad (2.2)$$

The difference between the theoretically derived expression and the empirically derived one rested on a factor,

$$\mu_g=\left(1-\frac{1}{4}\frac{v^2}{c^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

That is to say, if the theoretically derived formula, *on the basis of only electrical effects*, was to be multiplied by (2.3), then a correct expression,

$$q\equiv q_0\left(1-\frac{1}{3}\frac{v^2}{c^2}\right)\left(1-\frac{1}{4}\frac{v^2}{c^2}\right)$$

$$\equiv q_0\left(1-\cdot 58\frac{v^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

would be obtained. It will be remembered in the Heaviside method of analysis the electron charge q_0 was held fast in space, and the ætherial medium allowed to flow past with velocity v. No allowance was made for possible gravitational modifications, because of a reaction (by analogy Einsteinian) due to the mass m being really projected forwardly simultaneously with the charge q_0. It will first be shown that the factor μ_g of (2.3) is quite in accord with the Einsteinian hypothesis, and then more particularly with the evidence that a force Q exists of the nature outlined in Section 1.

Indeed, it will be noticed that the kinetic energy in Einstein's system is expressed by

$$\mathrm{T}=m_0c^2(1-\sqrt{1-\beta^2}), \qquad \beta^2=(v/c)^2. \quad . \quad (2.5)$$

If, now, we develop the radical, it follows that

$$1-\sqrt{1-\beta^2}\equiv\tfrac{1}{2}.\beta^2(1-\tfrac{1}{4}\beta^2). \quad . \quad . \quad . \quad (2.6)$$

Substituting (2.6) in (2.5) we have

$$\mathrm{T}\equiv\tfrac{1}{2}m_0v^2\left(1-\tfrac{1}{4}\frac{v^2}{c^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (2.7)$$

The factor $(1-1/4.v^2/c^2)$ is now seen to be in agreement with the μ_g of (2.3).

3. A particle with *Newtonian Mass* m at A (see fig. 3.1) follows a trajectory A′AA″, subject to forces,

$$-\frac{\mathrm{P}}{m}=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2,$$

$$-\frac{\mathrm{Q}}{m}=\frac{1}{r}\frac{d}{dt}\left(r^2.\frac{d\theta}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

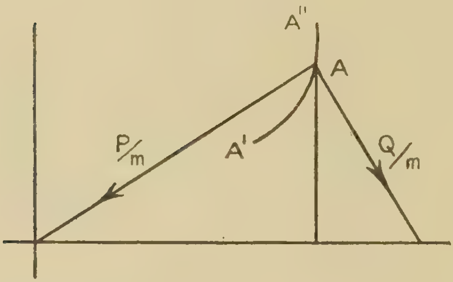
(see Phil. Mag., Jan. 1931). Carrying through the analysis obtaining for the "Dynamics of a Particle" (see Tait and Steele, *l. c.* p. 120, § 135), the expression h' in

$$h' = r^2 \frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

occurring in the Q equation of (3.1) becomes no longer a constant h characterizing the Keplerian rate with which equal areas are swept out by the radius vector in unit time. Instead, then, of the ordinary equation with $Q=0$ (following the method of T and S, *l. c.* p. 122), we have

$$\frac{d^2u}{dt^2} + u = \frac{1}{(h'u)^2} \cdot \left(\frac{P}{m} - \frac{dh'}{dt} \cdot \frac{du}{d\theta}\right) \quad . \quad . \quad . \quad (3.3)$$

Fig. 3.1.



But since it also follows that

$$\frac{dh'}{dt} = h'u^2 \cdot \frac{dh'}{du} \frac{du}{d\theta}, \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

the resulting equation takes the form

$$\frac{d^2u}{dt^2} + u = \frac{1}{(h'u)^2} \cdot \frac{P}{m} - \frac{1}{h'} \frac{dh'}{du} \left(\frac{du}{d\theta}\right)^2 \quad . \quad . \quad . \quad (3.5)$$

The latter constitutes the fundamental characterizing equation of the present non-time-space theory. The mass m is equally the Newtonian mass.

Incidentally, again, following the methods of Tait and Steele (*l. c.* pp. 124, 125, 129), it is easily proved that

$$\left(\frac{v}{h'}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{2} \frac{d}{du} \left(\frac{v}{h'}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (3.7)$$

Introducing (3.7) in (3.5), it is seen that

$$\frac{d}{du}\left(\frac{v}{h'}\right)^2=\frac{2}{h'^2}\cdot\frac{P}{mu^2}-\frac{2}{h'}\frac{dh'}{du}\left[\left(\frac{v}{h'}\right)^2-u^2\right], \quad . \quad (3.8)$$

so that by rearranging terms it is easily checked ;

$$\frac{dv^2}{du}=2\,.\frac{P}{mu^2}+2h'\,.\frac{dh'}{du}\,.u^2. \quad . \quad . \quad . \quad (3.9)$$

Integrating the equation we have (*l. c.*)

$$v^2=C+2\int\frac{P}{mu^2}du+2\int h'\,.\frac{dh'}{du}\,.u^2du. \quad . \quad (3.10)$$

4. The constant C in (3.10) can now be evaluated by regarding the conditions obtaining for extremely large values of r, implying thereby that then h' approaches the Keplerian value h. In this way (see *l. c.* p. 136) it follows instead of (3.10) that we have

$$v^2=\frac{2P}{mu^2}\left(u-\frac{1}{2a}\right)+2\int h'\frac{dh'}{du}u^2du. \quad . \quad . \quad . \quad (4.1)$$

The quantity $2a$ is the "major axis" of the ellipse. Factoring in (4.1) there results

$$v^2=\frac{2P}{mu^2}\left(u-\frac{1}{2a}\right)(1-X). \quad . \quad . \quad . \quad (4.2)$$

But, then, in accordance with the evaluation of C in (3.10) leading to (4.1), it is necessary to note by definition, we have

$$\frac{P}{u^2}\left(u-\frac{1}{2a}\right)=\tfrac{1}{2}mv_0{}^2=T_{dh'=0}. \quad . \quad . \quad . \quad (4.3)$$

The v_0 thus defines that value of v which would obtain had the extreme right-hand term of (3.10) been declared negligible. This simplifies matters considerably, for we have now from (4.1) and (4.2) that

$$v^2=v_0{}^2(1-X), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

$$X=-\frac{2}{v_0{}^2}\,.\int h'\,.\frac{dh'}{du}\,.u^2du. \quad . \quad . \quad . \quad (4.5)$$

A concordance can be effected with (2.7) if we introduce an "effective mass" m_e (somewhat after the Einsteinian manner), where we define

$$\tfrac{1}{2}mv_0{}^2=\tfrac{1}{2}m_ev^2, \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

This enables us to write, instead of (4.4),

$$T=\tfrac{1}{2}mv^2=\tfrac{1}{2}m_e v^2(1-X). \quad . \quad . \quad . \quad (4.7)$$

There arises now, as a result, two alternatives characterizing the "effective mass," namely,

$$m/m_e=(v/v_0)^2, \qquad m/m_e=1-X. \quad . \quad . \quad (4.8)$$

The second for the present purpose is the more useful.

In the Kaufmann-Bucherer experiments the gravitational factor indicated in (2.3) was

$$\mu_g=1-\frac{1}{4}\frac{v^2}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.9)$$

This was, however, for a substantially circular trajectory. For this circumstance, therefore, we have

$$v^2\equiv(h'u)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.10)$$

$$\mu_g\equiv 1-\frac{1}{4}\frac{h'^2u^2}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.11)$$

In view of (4.4), (4.7), and (4.5), it should then follow

$$X\equiv\frac{1}{4}\left(\frac{h'u}{c}\right)^2\equiv-\frac{2}{v_0{}^2}\cdot\int(\;)\,du$$

$$\equiv-\frac{2}{(h'u)^2}\cdot\int h'\cdot\frac{dh'}{du}\cdot u^2du. \quad . \quad . \quad (4.12)$$

It can now be checked that the solution of (4.12) is in agreement with the result

$$h'=h\left[1-\frac{1}{4}\frac{h^2}{c^2}\left(\frac{u^2-\left(\frac{u_a}{N}\right)^2}{1-1/N^2}\right)\right]. \quad . \quad . \quad (4.13)$$

The reason for the choice of constant indicated in (4.13) will better appear later. Suffice it to say at the moment that u_a is the constant value of u obtaining for a circular trajectory and u_a/N a consequent portion of u_a taken. Thus for circular paths—as well as for large values of N constant—the factor in (4.13) agrees with (4.11). For exceedingly large values of u relative to u_a the u^2 becomes dominant, and we rather have that

$$h'\to h\left(1-\frac{1}{4}\frac{h^2u^2/c^2}{1-1/N^2}\right). \quad . \quad . \quad . \quad (4.14)$$

The denominator $1-1/N^2$ becomes necessary if a correlation with Einstein's gravitational theory should be sought.

5. Turning back to the orbital equation (3.5), which is really given by

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{h'^2}\frac{\mathrm{P}}{mu^2}-\frac{1}{h'}\frac{dh'}{du}\left(\frac{du}{d\theta}\right)^2, \quad . \quad . \quad . \quad (5.1)$$

the latter can be solved in terms of u and θ only if something more is known regarding the laws of the Q-term,

$$-\frac{1}{h'}\frac{dh'}{du}\left(\frac{du}{d\theta}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad (5.2)$$

in its relation to the Newtonian function P. That the expression (5.2) cannot be entirely arbitrary would appear to be obvious. In 'Nature' some definite functional relationship must subsist between the Q of fig. 3.1 and the Newtonian force P at right angles thereto. Then, inasmuch as (5.1) can be solved by immediate integration with the $\left(\frac{du}{d\theta}\right)^2$ term missing, the possibility does arise that the right-hand side of (5.1) is already a complete differential. For, after all, we really have instead of (5.1) that

$$\tfrac{1}{2}\,.\,d\left(\frac{v}{h'}\right)^2=\frac{1}{h'^2}\frac{\mathrm{P}}{mu^2}\,du-\frac{1}{h'}\left(\frac{du}{d\theta}\right)^2.\,dh', \quad . \quad (5.3)$$

and two of the terms are perfect differentials. Euler's test can thus be employed.

The condition for a complete differential is that

$$\frac{\partial}{\partial h'}\left(\frac{1}{h'^2}\cdot\frac{\mathrm{P}}{mu^2}\right)_u=-\frac{\partial}{\partial u}\left(\frac{1}{h'}\cdot\left(\frac{du}{d\theta}\right)^2\right)_{h'}. \quad . \quad (5.4)$$

Performing the differentiation with respect to h', it follows that

$$\frac{1}{h'}\frac{\partial}{\partial u}\left(\frac{du}{d\theta}\right)^2\equiv\frac{2}{h'^3}\cdot\frac{\mathrm{P}}{mu^2}. \quad . \quad . \quad . \quad (5.5)$$

Then by partial integration with respect to u it is seen that

$$\left(\frac{du}{d\theta}\right)^2\equiv\frac{2u}{h'^2}\frac{\mathrm{P}}{mu^2}+\mathrm{H}'. \quad . \quad . \quad . \quad . \quad (5.6)$$

The Function H$'$*, however, must be independent of* u, though it can contain h' and for inverse square forces P the expression $\mathrm{P}/(mu^2)$.

Consideration now shows that since for circular trajectories $(du/d\theta)^2$ becomes zero, therefore H$'$ must involve only a set of discrete values, one for each discrete value to be given to

the possible radii. This is a very important conclusion, for by means of it a direct linking follows of the present central field mechanics based on the Newtonian concepts of time and space, with the laws of Quantum theory as originally outlined by Bohr for atomic systems.

By the very necessity of the case we have now, instead of (5.6),

$$\left(\frac{du}{d\theta}\right)^2 \equiv \frac{2}{h'^2}\frac{P}{mu^2}(u-u_a), \quad . \quad . \quad . \quad (5.7)$$

with u_a having a series of possible discrete values. Introducing (5.7) into (5.1), we have that

$$\frac{d^2u}{d\theta^2}+u \equiv \frac{1}{h'^2}\frac{P}{mu^2}\left[1-\frac{2}{h'}\frac{dh'}{du}(u-u_a)\right]. \quad . \quad (5.8)$$

6. Turning now to the gravitational evaluation (4.13), if the value of h' is introduced into the orbital factor of P in (5.8), we have, in the first place that we can set,

$$M_0 = \left(\frac{h}{h'}\right)^2 . \left[1-\frac{2}{h'}\frac{dh'}{du}(u-u_a)\right]. \quad . \quad . \quad (6.1)$$

Clearly from (4.13) we have

$$\left(\frac{h}{h'}\right)^2 \equiv 1+\frac{h^2}{2c^2}.\left(\frac{u^2-\left(\frac{u_a}{N}\right)^2}{1-1/N^2}\right). \quad . \quad . \quad (6.2)$$

On the other hand, in (6.1),

$$[\ ]\equiv 1+\frac{h^2}{c^2}\frac{u(u-u_a)}{1-1/N^2}. \quad . \quad . \quad . \quad . \quad (6.3)$$

Combining the last three equations, it is seen substantially that we have

$$M_0 \equiv 1+\frac{3}{2}\frac{h^2u^2}{c^2}.\frac{1}{1-1/N^2}-\frac{h^2}{c^2}\left[\frac{uu_a+\frac{1}{2}\left(\frac{u_a}{N}\right)^2}{1-1/N^2}\right]. \quad (6.4)$$

Provided, therefore, that extreme ellipticity does not occur, that is to say, u is never very materially different from u_a, we can write

$$M_0 \equiv \left(\frac{h}{h'}\right)^2 \equiv 1+\frac{1}{2}\frac{h^2u_a^2}{c^2}. \quad . \quad . \quad . \quad (6.5)$$

With sufficiently obvious evidences of ellipticity, where, over most of the trajectory

$$u >> u_a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.6)$$

we have from (6.4) that

$$M_0 \equiv 1 + \frac{3}{2}\frac{h^2u^2}{c^2} \cdot \frac{1}{1-1/N^2} . \quad . \quad . \quad . \quad (6.7)$$

Accord with Einstein's gravitational equation is then obtained by giving to N^2 the value *one-half*. It then follows that

$$M_0 \equiv 1 + 3\frac{h^2u^2}{c^2}, \quad . \quad . \quad . \quad . \quad . \quad (6.8)$$

and the orbital equation becomes

$$\frac{d^2u}{d\theta^2} + u \equiv \frac{1}{h^2} \cdot \frac{P}{mu^2} \cdot \left(1 + 3\frac{h^2u^2}{c^2}\right) . \quad . \quad . \quad (6.9)$$

The more general expression, however, is (5.8) in view of (4.13). Equation (5.8) then *gives better results than the Einsteinian* (6.9) *for the smaller eccentricities* of the Earth and Venus (see Eddington's 'Report on the Theory of Relativity and Gravitation,' p. 52 (1920); also his 'Mathematical Theory of Relativity,' pp. 89–90 (1923), where he states "the residual of the node of Venus is rather excessive .... and may perhaps be a genuine discordance.")

7. In the theory of atomic structure it becomes necessary to know the law of potential energy V as a function of the radial distance r of the electron. In (3.9) it was given as

$$\frac{dv^2}{du} = 2\frac{P}{mu^2} + 2h'\frac{dh'}{du} \cdot u^2 . \quad . \quad . \quad . \quad (7.1)$$

The kinetic energy T, however, in (4.7) was defined as

$$T = 1/2 \,.\, mv^2, \quad . \quad . \quad . \quad . \quad . \quad (7.2)$$

and was shown to be in accord substantially with the Relativistic viewpoint of Einstein. Thus we have indeed for the orbital rate of change

$$\frac{dT}{du} = \frac{d}{du}(1/2 \,.\, mv^2) = \frac{P}{u^2} \cdot \left[1 + \frac{mu^2}{P} \,.\, h'\frac{dh'}{du} \,.\, u^2\right] . \quad (7.3)$$

The factor [ ] in (7.3) can now be regarded as expressing the modification of P at a distance that arises because of the existence of a transverse force Q.

If, then, the system as a whole is conservative with value E, the potential energy V is given by

$$V = E - T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.4)$$

and therefore we have

$$\frac{d}{du}(\mathrm{E}-\mathrm{V})=-\frac{d\mathrm{V}}{du}=\frac{\mathrm{P}}{u^2}\cdot[\ ]. \quad . \quad . \quad . \quad (7.5)$$

(The E's, however, must be as discrete as the u_a's.) Inasmuch as u is defined by (3.1) it also follows that

$$\frac{d\mathrm{V}}{dr}=\mathrm{M}_1 \,.\, \mathrm{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.6)$$

$$\mathrm{M}_1=1+\frac{mu^2}{\mathrm{P}}\cdot h'\frac{dh'}{du}\cdot u^2. \quad . \quad . \quad . \quad (7.7)$$

The multiplying factor for the inverse square of force P is thus the *theoretically derived* quantity M_1. The factor becomes important in connexion with the inter-atomic force theory of Kratzer and the assumption of Sommerfeld for the hydrogenic intra-atomic fields.

To evaluate (7·7) we have from (4.13)

$$h'\equiv h\left[1-\frac{1}{2}\frac{h^2}{c^2}(u^2-\tfrac{1}{2}u_a{}^2)\right]. \quad . \quad . \quad . \quad (7.8)$$

It is then seen that

$$\frac{dh'}{du}\equiv-\frac{h^3u}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.9)$$

Substituting the latter in (7.7) we have

$$\mathrm{M}_1=1-\frac{mu^2}{\mathrm{P}}\frac{h^4u^3}{c^2}\left[1-\frac{1}{2}\frac{h^2}{c^2}(u^2-\tfrac{1}{2}u_a{}^2)\right]. \quad . \quad (7.10)$$

Rearranging terms and setting with Sommerfeld (see Ruark and Urey, 'Atoms, Molecules, and Quanta,' p. 197),

$$\mathrm{C}_3=-\frac{mu^2}{\mathrm{P}}\left[1+\frac{1}{4}\left(\frac{hu}{c}\right)^2\right]\cdot\frac{h^4}{c^2}, \quad . \quad . \quad (7.11)$$

$$\mathrm{C}_5=\frac{1}{2}\left(\frac{mu}{\mathrm{P}}\right)\cdot\frac{h^6}{c^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.12)$$

we have for the law of force,

$$\mathrm{F}=\frac{d\mathrm{V}}{dr}=\mathrm{P}\left(1-\mathrm{C}_3\frac{1}{r^3}+\mathrm{C}_5\cdot\frac{1}{r^5}\right). \quad . \quad . \quad (7.13)$$

The fact that the inverse square and inverse fourth powers of r are missing in M_1 is quite in keeping with the quoted observations of Born and Heisenberg "that the term containing the inverse fifth power is probably the most important except

the inverse square." The question of atomic "polarizability" thus takes on quite a new aspect.

Kratzer (see R. and U., *l. c.* p. 376), in connexion with the hydrogen halides, had already *assumed* a binding force between the two nuclei that involved the usual attractive force proportional to the inverse square of r (the P in (7.13)), and a repulsional force proportional to $1/r^3$ (see, for example, "Molecular Spectra in Gases," Bull. Nat. Res. Council, p. 299). This is clearly indicated in (7.13). The mystery of repulsional forces is thus found to be due to the possible existence of the transverse force Q at right angles to the Newtonian P.

December 1932.

XCIII. *Magnetic Structure of an Iron Alum in Strong Fields.* By J. Forrest, *M.A., D.Sc., Physics Dept., University College, Dundee* *.

SOME measurements † of the writer in magnetic fields of the order of 20,000 gauss on plates ground from crystals of iron ammonium alum

$$[Fe(SO_4)_3(NH_4)_2SO_4 . 24H_2O]$$

gave indications of anisotropy in these fields, and in particular of magnetic cubic symmetry. In that work on many specimens of this and other crystals of like type the occurrence of turning values at cubic intervals was of too regular incidence to be instrumental or accidental in origin. There was, however, lack of agreement in the incidence of the turning values of the two components of magnetization parallel and transverse to the applied field. The reasons for this want of correspondence were not at that time fully realized. The curves then given were not claimed to be of high accuracy as regards their form in view of the high sensitivity required in the measurements and the many sources of error involved in such measurements in high fields.

Recently C. G. Montgomerie ‡ has criticized these curves for the parallel component in this crystal. From

* Communicated by the Author.
† Trans. Roy. Soc. Edin. liv. (1926).
‡ Phys. Rev. 1930, Dec. 1.

his determinations of the parallel component in low or medium fields up to 5000 gauss by a "force" method, he concludes there is no definite evidence of anisotropy in these fields, and argues that in absence of variations in low fields in a paramagnetic crystal there cannot be variations in higher fields. This argument, while perhaps at first sight attractive, is untenable, for it postulates a perfect crystal, free from internal foliation and from internal strain, with a perfect boundary or no boundary at all. Instead of the transverse component being non-existent, measurements at a sufficient sensitivity indicate its existence, even in low fields, due in part at least to some of the above effects, and probably partly molecular in origin.

The purpose of this paper is to indicate the influences present in such crystals and to give an account of some re-measurements of the transverse component in principal planes in high fields.

§1. Generally three distinct effects are recognizable.

(*a*) First there is the intrinsic magnetic quality and any possible real variations of the same, while the orientation of the applied field varies within the specimen.

(*b*) Internal strain, which is present more or less in all crystals, has a strong influence in weakly magnetic cubic crystals in low or moderate fields, and tends to mask any real variations. The existence and importance of internal tension in such crystals is indicated by the anomalous interference figures given by transparent cubic crystals in polarized light. The existence of these figures with their variation from specimen to specimen of the same substance is taken as evidence of the presence of internal strains in such crystals.

Such internal strains give a pseudo-structure which leads to small variations of the parallel component of magnetisation, and hence also to a corresponding transverse component. The simplest type of strain is that giving a direction of easy magnetization with a perpendicular direction of more difficult magnetization, *i. e.*, the effect of a pseudo-orthorhombic structure. In the parallel component as determined by a force method the variations due to this structure may be comparable with the instrumental error or the probable error of measurement in low fields.

Hence such a strain effect may introduce a "$\cos 2\theta$" term into the curve of the parallel component and give a transverse component with minimum values at intervals of 90°.

Similar in effect to internal strain we generally have foliation along one definite cleavage plane or along several planes, leading to variations of the same type as in the case of strains.

The influences of internal strain and foliation are met with in weakly magnetic crystals belonging to other systems than the cubic system. In some cases the relative order of magnitude of the intensity of magnetization along the various principal axes is found to be different in low fields from the relative order for high fields in the same specimen.

§2. (*c*) Errors in the boundary of the specimen also tend to produce variations in the parallel component and to give also a corresponding unreal transverse component. If the circular disk form of the specimen is imperfect and contains one diameter slightly greater than the others, then the disk, if weakly magnetic, will exhibit in low fields a pseudo-orthorhombic structure. When freely suspended in the horizontal plane between the poles of a magnet, the maximum diameter of the disk tends to set parallel or transverse to the field, according as the substance is paramagnetic or diamagnetic. This is the position of maximum stability of such a disk in the field if there is no other direction of less potential energy in it. Such an effect will give again a quasi-orthorhombic structure and so an unreal variation which is not molecular in origin.

This source of error is a minimum for a disk thin compared with its diameter and a maximum for a cylinder with greater lateral surface on which such defects can occur. It is difficult to grind a small cylinder from a crystal with such accuracy that every element of area on its curved surface is exactly at the same distance from a common axis. By repeated checking and grinding of the surface or edge of the disk such discontinuities can be diminished, but in the case of weakly magnetic cubic crystals it is difficult to eliminate this effect altogether. If this source of error exists in a disk or cylinder made from a weakly magnetic cubic crystal,

the experimenter, when making magnetic measurements thereon, may be merely testing the accuracy with which the disk has been ground.

The same influences can be appreciable in any plane of high symmetry of a weakly magnetic crystal, as in the cubic plane of a tetragonal crystal, in the (1, 1, 1) plane of a trigonal crystal, or, in general, in a crystal plane in which there are two or more possible directions of equal or nearly equal maxima of intensity of magnetization, or a real variation of small amount.

The influence of one or other of these effects probably accounts, in some part at least, for the non-cubic components indicated by C. G. Montgomerie in the curves for the parallel component in iron ammonium alum. However, in view of the high sensitivity involved and the unavoidable sources of error the value of analysis of a single curve is doubtful. Such analysis to have any weight should be carried out on several curves for more than one specimen.

§3. It is clear that any real variations in such crystals cannot be evident with certainty in low or medium fields owing to the presence of the above effects, which may in their turn produce unreal variations. However, if the strength of the field be raised real effects may increase at a faster rate than unreal effects. Thus it is probably in general only in high fields that any real variations can be distinguished. The weaker the susceptibility the higher must be the applied field for diminution or elimination of the unreal effects. It appears, finally, that determinations of variations of magnetic quality of weakly magnetic cubic crystals can yield information of value generally only when carried out in high fields. Hence, viewed in this sense, there is a field effect even although the substance be paramagnetic or diamagnetic.

If the substance were a perfect crystal internally and the boundary perfectly circular or non-existent no such limitation would be necessary. Due to the imperfections it is generally difficult to give the experimental curves the accuracy and final completeness desirable in such measurements.

§4. In the measurement of the transverse component of magnetization in such crystals, there is another source

of error which is difficult to eliminate when the specimen has any lateral freedom of movement in the field. As the poles of the electromagnet are brought closer together for the production of high uniform fields, the suspension carrying the specimen must be very accurately centred between the poles and must so remain as the field is rotated in azimuth, in order to prevent a pull by either pole on the disk. Such a force on the specimen gives a false deflexion of the suspension which is not a true rotation. It is probably this source of error that predominates in the previous curves for the transverse component in this crystal, *i. e.*, iron ammonium alum. The same effect is experienced in measurements on more strongly magnetic (cubic) crystals, but not to so great an extent as in the case of weakly magnetic cubic crystals, for in order to obtain the higher fields required in the latter case the magnet-poles must approach closer to the specimen.

For a strongly magnetic crystal this tendency can be overcome by loading of the comparatively robust suspension or by double mooring of the suspension. In the case of weakly magnetic crystals the latter procedure is alone available, as loading of the suspension is limited by the fineness of the control fibre necessary for measurement of comparatively weak couples.

Any method of measurement of either component has its own peculiar sources of error which are intensified in effect by the high sensitivity required. Even in the same field the curves for the two components may not agree completely unless the sources of error can be eliminated or distinguished. Such inconsistency is, of course, due to causes external to the specimen and inherent in the methods of measurement.

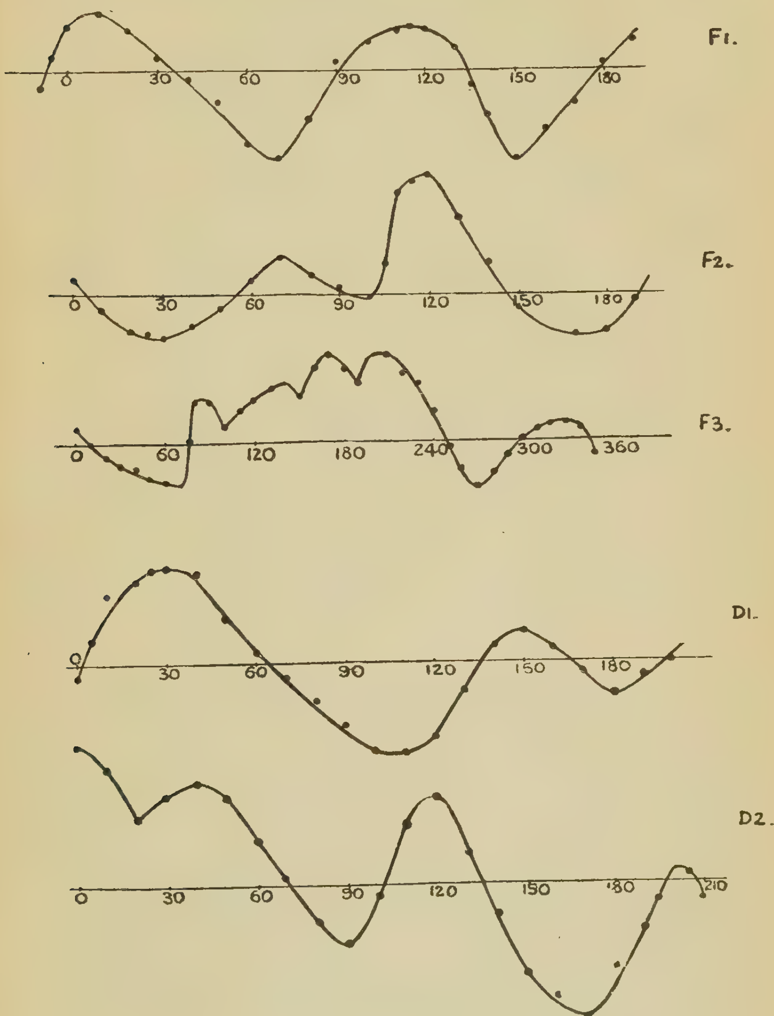
§5. The following measurements of the transverse component in ammonium iron alum crystals were made with a water-cooled electromagnet of the Weiss type. Since each reading only occupies a few seconds, it was possible, by overloading this magnet under proper precautions, to attain a field of approximately 30,000 gauss in a pole gap of 4 mm. Most of the measurements were made in a field of approximately 25,000 gauss, and then some final readings were taken in the stronger field.

Several suspensions were used, but the form of each was the same, consisting of two lengths of copper wire of S.W.G. Nos. 36, 40, or 44 twisted tightly together, and giving a flexed length of 80 cm., with a small loop left open at a distance of 27 cm. from one end. This flex was lightly soldered up to the loop, and two thin copper foil vanes were soldered to the end farthest from the loop. The flex formed the central part of a double-moored suspension held between two fine phosphor-bronze wires of diameter ·0025 cm., each generally 10 cm. long. The lower fibre was attached to a lead weight placed in a vessel containing the damping liquid, so that the lower fibre, the damping vanes, and a few centimetres of the lower end of the flex were within the liquid. A small mirror attached to the top end of the flex with a touch of fixative was of the smallest possible mass and dimensions, being about 3 mm. long by 2 mm. broad. The small crystal disk was inserted in the small loop in the flex, and was held in position by tension applied at the upper grip, which held the top end of the upper fibre.

The above type of suspension when under tension is quite straight, and the flex is rigid to small couples. In addition, it has the necessary property that couples due to its magnetic quality are negligible in high fields compared with those due to the crystal. In addition, the tension applied reduces to a minimum, and practically prevents the action of, any lateral pull on the crystal when the field is applied. Initially there is a drift of the zero, but in time this ceases when the suspension acquires permanent set. The arrangement, of course, has the defect that removal of the crystal alters the rigidity.

The deflexions due to these suspensions alone, under the most favourable circumstances for a maximum, were never greater than 2 mm. on a scale at a distance of 1·25 metres, while the maximum deflexion when the crystal disk was inserted in the stirrup was of the order of 7 cm. Hence, the couples due to the suspension when present only slightly affect the occurrence of zero values in the couple due to the crystal and alter the magnitude of the latter by a relatively small amount. The nature of the main effect is not altered. The deflexions were purposely kept small in order to diminish any tendency for lateral movement when the specimen rotated.

The disks were ground from the crystals with fine glass paper. As these disks had to be small in size, special small grips and guides were constructed to ensure



accurate grinding and minimum handling of the crystal. The maximum error in the inclination of the plane of a disk is probably within 5° to 10°.

§6. The curves show the variation of the transverse couple as the field is rotated in azimuth by steps of 10° (or 5°) in the plane of the base of the crystal disk or cylinder. Curves of type F are for face planes (1, 0, 0) and those of type D are for diagonal planes (1, 1, 0) of the cube. The curves are drawn to arbitrary scales, but the order of magnitude of these couples is given below. Since the angle of rotation of the specimens is of very small magnitude, these curves also practically indicate variation of the transverse component of magnetization.

The positions of the crystal axes are not indicated in these curves, as in order not to introduce any discontinuity on the surface of the small disks it appeared inadvisable at this stage to make any mark whatever on them. A reference mark on such a small disk may give rise to unreal couples in high fields.

The dimensions of disk F(1) were: diam. ·285 cm., thickness ·175 cm., and of D(1), diam. ·289 cm., thickness ·237 cm., the others being of somewhat similar size.

Curves F 1 and D 1 are typical of curves obtained, with suspensions of different sensitivities in the highest field used, about 30,000 gauss, for disks ground from several crystals. While slight differences are found from specimen to specimen in the relative values of the maxima and in the intervals of separation of the zero values of the couple, the main effect is always distinct.

§7. Many specimens were dealt with for which the curves of type F (1) and D(1) were not found. The curves for variation of the transverse component are of the type shown in curves F(2), F(3), and D(2). In these, however, the intervals of separation of maxima and minima differ little, within the limits of experimental error, from those in curves F(1) and D(1). I incline to interpret curves like F(2), F(3), and D(2) as showing the superposition of the effects of internal strain and foliation on a real cubic variation. This crystal has a primary cleavage along the (1, 1, 1) plane and a secondary cleavage parallel to the (1, 1, 0) plane. It is, of course, possible that the superposed variation may be partly molecular in origin.

In other cases the intervals of separation of maxima and minima were found to be slightly different from those indicated above. This was generally found to be due

to inaccurate setting of the disk in the stirrup or between the magnet-poles. Other cases could be ascribed to undue error in grinding the plane of the disk from the crystal, or in setting that plane horizontally in the stirrup.

In lower fields the minimum values of the couple occur apparently more or less at random intervals, with occasional partial appearance of cubic intervals. It is only as the field is raised that the latter variations tend to predominate, though not always at the same field for different specimens. Hence the curves F 1 and D 1 probably also contain the effects of unreal variations, though to a less degree than the other curves shown.

It is probable that crystalline irregularities must play a considerable part in masking the true or molecular variations. Discontinuities within the disk will have a greater effect in weakly magnetic substances than in more strongly magnetic substances. In this connexion an analogy may be found in the work of Weiss on pyrrhotine, for which true variations were found only by subdivisions of the crystals for isolation of a small perfect element.

To satisfy theories which demand an absolute isotropy at all fields, there may be an inclination to ascribe the above effects wholly to some of the influences indicated at the commencement of this paper. In the opinion of the writer this possibility is discounted by the more or less regular occurrence of the effect from specimen to specimen.

§8. Direct determinations of the sensitivities of the suspensions when under tension indicate that these couples correspond to a transverse susceptibility of the order of 10^{-7} to 10^{-8} c.g.s. units. This, when compared with a mean susceptibility of order 10^{-6} c.g.s., appears to give a rather smaller inclination of the magnetization to the field than the previous curves for the parallel component made by a force method indicate. It is, of course, generally of doubtful value to co-relate directional variations with the mean susceptibility. Since in the measurement of the parallel component by a force method for such specimens in high fields, it is generally difficult to keep the field gradient sufficiently small, it is probable that further measurements of both components in high uniform fields for more than one specimen are necessary

before an accurate estimate of the inclination of the magnetization to the field can be found.

Contrary to recent arguments *, Thomson's (Kelvin's) theory does not "predict," but, by its initial assumptions and mode of formulation, clearly *assumes* that all cubic crystals without exception are magnetically isotropic. Hence, without in any way denying to that theory its correct place in the development of the subject of molecular magnetism, it is difficult to see how it can be put forward now as an argument against these effects.

I hope to publish at a later date further and more quantitative work on crystals of this substance and on others of like type.

XCIV. *The Spectra emitted by High-Frequency and Direct Current Discharges in Helium.* *By* J. E. KEYSTON, *Magdalen College, Oxford* †.

[Plates XXI. & XXII.]

1. A NUMBER of experiments have recently been made at the Electrical Laboratory, Oxford, to determine the energies of the electrons that excite the light in the uniform luminous columns of discharges in wide tubes. Helium is of particular interest, since remarkable changes in the colour of the discharge are obtained in high-frequency discharges by changing the pressure of the gas or the distance between the electrodes. Also the principal lines of the visible spectrum of helium lie so far apart from one another on the wave-length scale that a single line may be separated from the remainder by means of a suitably chosen colour filter.

It was therefore possible to measure the intensities of the most prominent visible lines of the helium spectrum and of the continuous helium spectrum with the aid of filters. A general description of the colour of the light emitted by discharges in helium at various pressures has been given by Townsend and McCallum.

This paper describes the results of a spectrographic investigation of the light emitted by helium discharges in

* L. W. McKeehan, Reviews of Modern Physics, ii. Oct. 1930, p. 486.
† Communicated by Prof. J. S. Townsend, F.R.S.

the visible and ultra-violet regions. Photographs were taken of the light from different parts of direct-current and high-frequency discharges. The instrument used was a Bellingham-Stanley No. 2 Spectrograph. The discharge-tubes were of quartz about 4 cm. in diameter which contained helium at various pressures from 1 mm. to 57 mm. Usually the currents in the discharges did not exceed 10 milliamps. The methods used for purification of the gas and for excitation of the discharges were the same as described in previous papers.

Of the two principal bright regions in direct-current discharges, the positive column and the cathode glow, special interest attaches to the former, as a comparatively simple theory has been found * which accounts for the phenomena observed in this part of the tube. The theory depends on the results of determinations which were previously made of the mean energies of agitation of the electrons in uniform electric fields.

Three types of spectra are emitted from the positive column of helium—they are the line spectrum, the band spectrum, and a continuous spectrum. The principal properties of these spectra are as follows.

2. *The Line Spectrum.*—There are two types of line series, the series of triplet lines and the series of singlet lines. For a given current all lines of these series increase in intensity as the gas pressure is reduced from 57 mm. to 1 mm. It has long been known that the variation of intensity with pressure is widely different for the triplet and singlet lines. Direct quantitative measurements of the variations of the intensities of the light from the positive column have been made by Townsend and Jones † by means of colour filters and photo-electric cells, and they found wide differences between the variations of intensity with pressure for the yellow triplet line $\lambda 5876 (2^3P-3^3D)$ and the green singlet line $\lambda 5016 (2^1S-3^1P)$. The variations with pressure of these two lines can be clearly distinguished in the reproductions of spectra, shown in fig. 1 (Pl. XXI.), of the emission from the positive column in tubes containing helium at different pressures. The

* J. S. Townsend, Phil. Mag. ix. p. 1145 (1930); xi. p. 1112 (1931); xii. p. 745 (1932).

† J. S. Townsend and F. Ll. Jones, Phil. Mag. xii. p. 815 (1931).

time of exposure was the same in each case (15 minutes), and the current was also kept constant (10 milliamps). At the highest pressure, 57 mm., the only prominent lines appearing on the photographs are λλ5876(2^3P-3^3D), 4471(2^3P-4^3D), 3889(2^3S-3^3P), and 3188(2^3S-4^3P), the lower members of the first subsidiary and principal triplet series of helium: the first line gives the characteristic yellow appearance to the positive column at high pressures. As the pressure is reduced from 57 mm to 1 mm. the higher members of the triplet series appear together with the lower members of the singlet series. Further decrease of pressure is accompanied by an extremely rapid increase in intensity of the singlet lines with respect to the triplet lines, and at pressures less than 1 mm. the column acquires a green colour due to the great intensity of the line 5016(2^1S-3^1P). It must be remembered that the reproduced photographs give no indication of the true relative intensities of the yellow and green lines, since the photographic plates which were used had a minimum of sensitivity in the green region; for example, the green line is absent from the photograph at 57 mm. but at this pressure it can still be easily observed with a direct-vision spectroscope.

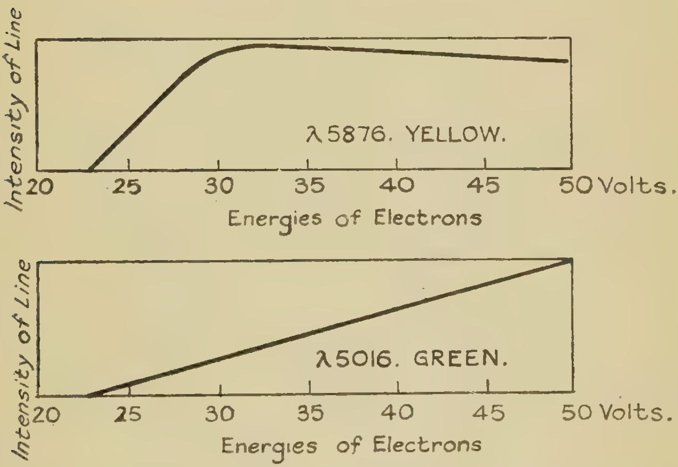
3. In our theory of the light emitted from the positive column of discharges in gases at high pressures, the excitation of the lines in the spectrum is attributed to single impacts of the electrons with atoms of the gas. The mean energy of agitation of the electrons is of the order of three or four volts, but the energies are distributed about the mean energy and some of the electrons attain high energies (greater than 20 volts), which are sufficient to excite the spectral lines or to ionize the normal atoms by single collisions.

In helium at 23 mm. pressure, in tubes about 3·5 mm. in diameter, the mean energy of agitation of the electrons in the uniform column of the discharge is 2·8 volts, and the number of electrons that attain energies above 20 volts is *very much* less than the number in the discharge at 3 mm. pressure, where the mean energy is 3·9 volts. Thus, although the number of collisions of electrons with atoms is greater at the higher pressures than at the low pressures with the same current flowing in the gas, the intensity of

the lines is less at the high pressure than at the low pressures,

When the pressure is reduced to about half a millimetre there is a large increase in the number of electrons with large energies, and it is at these pressures where the green line become so intense in comparison with the yellow line that the discharge assumes a green colour. The increase in the intensity of the green line in comparison with that of the yellow is due to the increase in the energy of agitation of the electrons.

Fig. 2.



4. This result is in general agreement with the experiments on the intensities of spectral lines made by Hanle *, Michels †, and Lees ‡. In their experiments electrons with known velocities were projected into a space containing a gas at a low pressure (less than one-tenth mm.), and the intensities of the spectral lines due to the direct collision of the electrons with normal atoms was measured. The results of Hanle's experiments for the yellow and green lines of helium are shown in fig. 2 for velocities of electrons from 20 to 50 volts. The maximum intensity

* W. Hanle, *Zeits. f. Physik*, lvi. p. 94 (1929).
† W. C. Michels, Phys. Rev. xxxvi. p. 1363 (1930).
‡ J. H. Lees, Proc. Roy. Soc. cxxxvii. p. 173 (1932).

of the yellow line is obtained at about 32 volts, and the maximum for the green line is at about 110 volts. In general the maxima of the intensities of the triplet lines lie at much lower electronic velocities than the maxima of the singlet lines. It is clear from the curves that any large increase in the number of electrons in the discharge with velocities greater than 32 volts will lead to an increase in the intensity of the green line λ 5016 as compared with the yellow line λ 5876, and a gradual change in the colour of the uniform column from yellow to green may be expected as the pressure is reduced from a large value to about 0·5 mm. The general increase in the intensity of singlet lines with respect to triplet lines as the pressure is reduced is explained in a similar manner.

5. *The Band Spectrum.*—The band spectrum has been observed in the uniform column in helium at high pressures, but as the pressure is decreased the bands decrease in intensity in comparison with the lines, and at pressures less than 10 mm. they become very faint.

The most prominent bands in the positive column are the lowest bands of the triplet band series. These are the bands with heads at $\lambda\lambda$ 6398, ca. 5900, 5570, 4650, 4546, ca. 4450, and 3680 Å.

The intensities of all these bands increase with pressure from 1 mm. to 57 mm. ; the increase in intensity of the bands 6400, 4650, and 3680 can be clearly seen in fig. 1 ; the bands appear unresolved as wide blurred lines as a wide slit was used. No singlet bands have been observed in the positive column at any pressure between 1 mm. and 57 mm.

This increase of intensity of the band spectrum with the pressure is not reconcilable with the hypothesis that the band spectrum is emitted by helium molecules which are formed by the union of metastable and normal helium atoms.

It is supposed that metastable atoms are generated by electrons with energies of about 20 volts, and the yellow and green lines by electrons with energies of about 23 volts, so that the rate at which the metastable atoms are generated should, according to our theory, diminish with the pressure much the same as the rate at which the intensity of the yellow or green light diminishes. The

decrease of intensity with increase of pressure of the yellow and green helium lines as measured by Townsend and Jones is given in the following table :—

Pressure.	Yellow.	Green.
38·0 mm. ..	10	10
13·0 ,, ...	36	16
3·7 ,, ...	85	34
1·0 ,, ...	127	107

It can be seen that the intensity of the yellow and green light decreases roughly by one-tenth as the pressure is increased from 1 mm. to 38 mm. Hence the number of metastable atoms in the gas, and consequently also the number of molecules, should diminish in about the same ratio as the pressure is increased from 1 mm. to 38 mm. if the molecules are formed by the combination of a metastable atom with a normal atom. Instead of the observed increase in intensity of the bands with increase of pressure a considerable decrease in intensity would therefore be expected. Thus it cannot be supposed that the molecules which emit the band spectrum are produced by the combination of metastable atoms with normal atoms. The molecules must be initially present in the gas before a current is passed through it.

6. Widely different views as to the importance of metastable atoms have been expressed by different investigators. A well-known example is comprised in a theory of the development of the currents in gases, given by Franck and Hertz, where the ionization is attributed to the action of metastable atoms, as described in a paper by Atkinson *. According to this theory electrons which attain energies of 20 volts lose all their energy in collisions with atoms, and metastable atoms are generated. In gases at high pressure, where the ratio of the force X to the pressure p is small, as in these experiments, practically none of the electrons could attain

* R. d'E. Atkinson, Proc. Roy. Soc. A, cxix. p. 335 (1928).

energies above 20 volts, as they make so many collisions in which they lose the amount corresponding to 19·77 volts.

It has been suggested * that, in order to excite the spectral lines or to obtain ionization in a pure gas, it would be necessary to have a second collision in which the metastable atom acquires an additional amount of energy. Under these conditions the intensity of the spectral lines and the rate of ionization of the gas would be proportional to the square of the current, and the force required to maintain a current would diminish as the current increased. As the experiments with small currents show that the force is independent of the current and the intensities of the lines are directly proportional to the current, this theory must be rejected.

It is also stated that a large proportion of the metastable atoms set free electrons from metal surfaces †. On the other hand, Smith ‡, who has investigated the ionization of helium and other gases by the impacts of electrons with different velocities, does not attribute any importance to effects of metastable atoms on the electrodes.

It is therefore very doubtful to what extent metastable atoms may be generated in the gas, and the effect they may have on the conductivities of the gas or on the spectrum.

7. *The Continuous Spectrum.*—Besides the line and band spectra a continuous spectrum is emitted from the positive column of a helium discharge. This spectrum extends over the whole wave-length range of the spectrograph used in this investigation, *i.e.*, from 7500 Å. to 2200 Å., and most probably exceeds these limits in both directions. The intensity of the continuous spectrum increases as the pressure is increased from 1 mm. to 57 mm., as is seen in fig. 1 (Pl. XXI.). The apparent minimum of intensity in the region 5000 Å. to 6000 Å. in the photographs is due to the relative insensitivity of the photographic plates in this region.

This spectrum has been previously observed by several investigators, and its origin has been the subject of much

* S. P. McCallum and F. Ll. Jones, Phil. Mag. xii. p. 384 (August 1931).

† M. L. Oliphant, Proc. Roy. Soc. cxxiv. p. 228 (1929).

‡ P. T. Smith, Phys. Rev. xxxvi. p. 1293 (1930).

speculation. It has been frequently assumed that it is a recombination spectrum and that its appearance is a proof that considerable recombination of electrons and positive ions takes place in the positive column. Several objections to this hypothesis have already been mentioned in a paper by Townsend and Pakkala *, but there are also other points which are of interest.

As a recombination spectrum the continuous light should be limited on the long-wave side by the line-series limits. On this hypothesis the recombination spectrum would not extend into the visible region. Further, when a recombination spectrum appears the higher members of the line-series should appear with unusually large iutensities compared with the lower members of the series. The reverse of this is actually observed, for under those conditions most favourable to the appearance of the continuous spectrum only the two or three lowest members of the line-series can be seen above the continuous background.

Further evidence against a recombination theory of this spectrum is obtained from measurements of the variation of its intensity with current strength. These measurements have been made by Townsend and Pakkala by a filter method for that part of the continuous spectrum lying between 5000 Å. and 5700 Å., they find that the intensity of the continuous spectrum increases directly as the current strength whereas a recombination spectrum must vary in intensity as the square of the current. The same result is shown by photographic measurements of the intensity of the continuous spectrum at 4100 Å. and 3800 Å. Two equal exposures of the helium discharge at 27 mm. were taken on the same plate with currents of 5 ma. and 10 ma. Since it has been well established by measurements made in this laboratory with photo-cells that the intensities of the helium lines in the positive column at low pressures vary directly as the current strengths, a simple intensity scale could be impressed on the same plate by photographing the positive column of a discharge in helium at 2·5 mm. pressure with currents of 1·5, 4, and 7 milliamperes. The lines at λ 4120 Å. and 3819 Å. provided a scale for measuring the intensities of the continuous spectrum at λ 4100 Å. and λ 3800 Å. The

* J. S. Townsend and M. Pakkala, Phil. Mag. xiv. p. 418 (Sept. 1932).

following results were obtained with helium at a pressure of 27 mm.:—

Current.	Intensity.	
	λ 3800.	λ 4100.
5 ma.	1	1
10 ma.	1·8	2·1

These measurements, althongh of no high degree of accuracy, can be regarded as affording good confirmation of the observation of Townsend and Pakkala that the continuous spectrum varies approximately as the first power of the current, when the current is small.

For these reasons it cannot be maintained that the continuous spectrum in helium is due to recombination.

8. According to another theory a continuous spectrum may be excited in certain diatomic gases when the molecules are dissociated. In the case of helium it is almost certain from the following considerations that the continuous spectrum is not produced by the dissociation of molecules.

(*a*) Not only helium, but also neon * and argon † emit strong continuous spectra from the uniform column, and, as in the case of helium, these continuous spectra increase in intensity as the pressure increases. Any explanation of the continuous spectrum of helium must be applicable also to the continuous spectra of neon and argon. But neon and argon do not exhibit any band spectra, and therefore presumably do not form molecules. The continuous spectra of the inert gases cannot then be attributed to any molecular process. In the case of argon the continuous spectrum is much more intense with respect to the line spectrum at any particular pressure than in the case of helium.

(*b*) In the cathode glow of the direct-current discharge the band spectrum of helium appears with great intensity but is not accompanied by the appearance of the continuous spectrum. Similarly, in the high-frequency

* P. Johnson, Phil. Mag. xiii. p. 487 (1932).

† S. P. McCallum, L. Klatzow, and J. E. Keyston, 'Nature,' cxxx. p. 810 (1932).

discharge a strong band spectrum may be observed at both electrodes, but the continuous spectrum is absent. At a pressure of about 3 mm. the "purple" glow, which is characteristic of a strong band spectrum, can be made to fill the whole space between the electrodes, but no continuous spectrum is emitted.

(*c*) In the direct-current discharge at pressures below about 20 mm. there is a very rapid change of intensity in a short distance in passing from the Faraday Dark Space to the positive column, *i.e.*, the illumination from the column does not gradually decrease on the dark-space side but has an abrupt end. At pressures above about 20 mm. a green glow appears beyond the end of the yellow positive column and extends one or two centimetres into the Faraday Dark Space. This green glow consists almost entirely of continuous radiation, and it increases in intensity as the pressure increases. Photographs of the green glow at 37 mm. and 57 mm. pressure are shown in fig. 3 (Pl. XXII.); the current and time of exposure were the same in each case. The faint traces of lines which appear in the photograph are in all probability due to reflexion of light from the positive column and the cathode glow by the walls of the tube. Thus in the green glow the continuous spectrum is emitted without the band spectrum. One may conclude from the presence of the continuous spectrum between the Faraday Dark Space and the positive column that this spectrum is excited by electrons whose velocities are not sufficiently high to excite the line and band spectra.

9. The most reasonable conclusion to be drawn from the preceding observations is that the continuous spectra which are emitted by the uniform columns of inert gas discharges are produced by excitation of the atom by electrons whose velocities are considerably lower than the minimum velocities required for excitation of the line and band spectra. According to the spectroscopic model of the helium atom it is not possible for electrons to suffer inelastic energy losses in helium in amounts smaller than 19·77 volts, which is the lowest excitation potential of the atom. But the experiments of Townsend and Bailey * on the diffusion of electrons in helium cannot be explained on this hypothesis and definitely require

* J. S. Townsend and V. A. Bailey, Phil. Mag. xlvi. p. 657 (1923).

that electrons may suffer inelastic losses of energy in helium before they acquire an energy of 19·77 volts. The recent experiments of Van Atta * on the excitation potentials of helium and neon provide further evidence that inelastic losses of energy occur before the electrons acquire energies of 19·77 and 16·54 volts respectively. Continuous helium spectra have been observed in the extreme ultra-violet (λ 500 Å. to λ 1000 Å.) by Hopfield †. It is noteworthy that Hopfield was unable to explain the appearance of one of the two continuous spectra which he observed on the basis of the existing theory of the helium molecule. He was forced to postulate the existence of a new type of helium molecule differing widely from the existing model in the energy of its lowest state.

10. *The Cathode Glow.*—The spectrum of the cathode glow was also investigated. In a direct-current discharge in helium at pressures greater than 3 mm., the glow appears as two distinct disks of differently coloured light; in the high-frequency discharge in wide tubes these differently coloured zones appear as concentric cylinders at both electrodes.

It has been previously observed ‡ with a direct-vision spectroscope that, in the part of the electrode glow which faces the Faraday Dark Space, the band spectrum is exceptionally intense compared with the line spectrum and gives the glow a characteristic purple colour. The part of the glow which faces the electrode emits both a strong line spectrum and a strong band spectrum. The two parts of the glow will therefore be referred to as the line zone (facing the cathode), and the band zone (facing the Faraday Dark Space).

11. The photographs show that in the line zone both the line and the band spectra are very intense. At pressures greater than 1 mm. there is very little variation of the line spectrum with the pressure, which is probably due to the fact that the radiation is excited by electrons with the same high velocities at all pressures. At pressures less than 1 mm. the singlet lines appear very intense with respect to the triplet lines, the glow assumes the vivid

* L. C. Van Atta, Phys. Rev. xxxviii. p. 876 (1931).
† J. J. Hopfield, Astrophys. J. lxxii. p. 133 (1930).
‡ J. S. Townsend and S. P. McCallum, Phil. Mag. xii. p. 1168 (1931).

green colour of the line λ 5016, and lines of the helium spark spectrum can be observed. The line series of the arc spectrum can be followed up to very high members.

In the band zone the band spectrum is about as intense as it is in the line zone, but the line spectrum is less than one-tenth as intense as it is in the line zone. The spectra of the two zones of the cathode glow, with the gas at 37 mm. pressure, are shown in fig. 4 (Pl. XXII.). The two photographs were taken on the same plate, the currents and the times of exposure being the same. They show clearly that the lines are much more intense in the zone facing the cathode than in the zone facing the Faraday Dark Space.

12. A double cathode glow has been observed previously by Seeliger and Mierdel * in mixtures of helium with argon, neon, and mercury, and is apparently a general property of a discharge in a mixture of gases with very different excitation potentials. They pointed out that the part of the glow facing the Faraday Dark Space emits only the lines of the gas which has the lower excitation potential whilst the part facing the cathode emits the lines of both gases. If, as they suggest, the difference in the excitation potentials of the two gases is the direct cause of the appearance of a double cathode glow, it must be concluded from the appearance of the double cathode glow in pure helium that the helium band spectrum has a much lower excitation potential than the helium line spectrum.

All theories of the excitation of the line and band spectra in the cathode region must be highly speculative on account of the complications introduced by the large concentration of electrons and positive ions in this region.

The absence, in the range λ 7000 Å. to λ 2200 Å. of the continuous spectrum throughout the whole of the cathode glow is noteworthy.

I wish to thank Professor J. S. Townsend for much valuable help and advice throughout the course of this investigation, and Professor Soddy and Mr. Brewer for the loan of the spectrograph. I am indebted to the Department of Scientific and Industrial Research for a Senior Research Award which has enabled me to carry out this investigation.

* R. Seeliger and G. Mierdel, *Zeits. f. Physik*, xix. p. 230 (1923).

XCV. *A Simple Method for the Numerical Solution of Differential Equations: Note on Error and its Avoidance.*

By W. G. Bickley, *D.Sc., Imperial College of Science and Technology* *.

ONE of the most unsatisfactory features of the numerical integration of differential equations is the difficulty of assessing the error committed † or of assuring that it shall not affect the last figure it is desired to retain. True, the errors of the various integration formulæ are known, but are given in terms of differences (or differential coefficients) of a highish order ‡, and, especially when a simple integration formula is being used, the labour of computing these may even exceed that of the process proper. Recently § I gave a method which is simple in principle and convenient and speedy in use. Although some attempt was made to estimate the error, or at least to determine the interval so that the error should not be appreciable, the above still remains true, since the "error" term involves the fourth (total) differential coefficient. A much simpler method of testing the process at any stage has been discovered, involving the differencing of small numbers only, and that only to the second order; this enables errors to be detected and corrected at once, or makes the necessity of reducing the interval clear.

With the notation of the previous paper, the differential equation being

$$dy/dx=f(x,y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

y being successively determined at x_0, x_0+h, ... x_0+nh, using Simpson's Rule as the integration formula, *i. e.*,

$$y_n=y_{n-2}+h(f_{n-2}+4f_{n-1}+f_n)/3, \quad . \quad . \quad . \quad (2)$$

the error of a double step is $2h^5 f^{\mathrm{iv}}_{n-1}/180$ or $2h\Delta^4 f/180$. Imagine a horizontal difference table constructed for f_n. The fundamental interpolation formula is then

$$f_{n+r}=(1-\Delta)^{-r}.f_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* Communicated by the Author.

† H. Levy, Lond. Math. Soc. Journal, vii. p. 305 (1932).

‡ Levy, *loc. cit.* Also W. E. Milne, Amer. Math. Soc. Monthly, xxxiii. p. 455 (1926); J. R. Scarborough, 'Numerical Mathematical Analysis,' p. 155 (1930).

§ W. G. Bickley, Phil. Mag. (7) xiii. p. 1003 (1932).

Consequently,

$$
\begin{aligned}
y_n - y_{n-2} &= \mathrm{I}_n \\
&= \int_{(n-2)h}^{nh} f_{n+r}\,dx \\
&= h\int_{-2}^{0} (1-\Delta)^{-r} \,.\, f_n\,dr \\
&= h\,\frac{1-(1-\Delta)^2}{-\log(1-\Delta)}\,.f_n \\
&= 2h\left(1-\Delta+\frac{1}{6}\Delta^2-\frac{1}{180}\Delta^4-\frac{1}{180}\Delta^5\ldots\right)f_n. \qquad (4)
\end{aligned}
$$

The approximate value given by Simpson's Rule is

$$\mathrm{I}'_n = 2h(1-\Delta+\frac{1}{6}\Delta^2)f_n. \quad . \quad . \quad . \quad . \qquad (5)$$

The point of this note is that we may "short-cut" the determination of $h\Delta^4 f_n$ by the use of (4) and (5). Writing the error of Simpson's Rule as E_n, we have

$$
\begin{aligned}
\mathrm{E}_n &= \mathrm{I}_n - \mathrm{I}'_n \\
&= -h/90(\Delta^4+\Delta^5\ldots)f_n. \quad . \quad . \quad . \qquad (6)
\end{aligned}
$$

Rewriting (5) in the form

$$\mathrm{I}'_n = 2h(f_{n-1}+\frac{1}{6}\Delta^2 f_n), \quad . \quad . \quad . \quad . \qquad (5\,a)$$

and using the notation

$$\mathrm{D}_n = \mathrm{I}'_n - 2hf_{n-1}, \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

we have

$$\mathrm{D}_n = \tfrac{1}{3}h \,.\, \Delta^2 f_n ; \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

so that

$$\mathrm{E}_n = -(\Delta^2+\Delta^3\ldots)\mathrm{D}_n/30. \quad . \quad . \quad . \quad . \qquad (9)$$

For many purposes it is sufficient to take E_n as $-\Delta^2\mathrm{D}_n/30$, but if more accuracy is required (9) is equivalent to

$$\mathrm{E}_n = -(2\Delta^2\mathrm{D}_n - \Delta^2\mathrm{D}_{n-1})/30 \quad . \quad . \quad . \qquad (9\,a)$$

To calculate and note D_n at every step involves comparatively little extra labour. D_n seldom exceeds three figures, and in any case its regular variation is a check upon the calculation of f_n and I'_n. It will usually be obvious, without writing down differences, whether the error E_n is appreciable. If it is, and amounts to more than one- or two-tenths of a unit in the final place ultimately to be kept, it is safer to reduce the interval at

once. In this type of work it is false economy to attempt to use too large an interval.

An example has been manufactured to illustrate the use of this correction and to show that by its aid correct results can be obtained over a considerable range, using a large interval. This example is not, however, to be taken as quite typical, since in practice so large an interval would not be used if results of the order of accuracy shown below were aimed at. In actual practice, then, D_n would be much smaller.

In the table below we show how the correction reproduces values of e^x correct to five places, by considering e^x as an integral of the equation

$$dy/dx=y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

x.	dy/dx.	δy.	D_n.	ΔD_n.	$\Delta^2 D_n$.	E_n.	y_n.	ΣE.
0·0	1·00000	—	—	—	—	—	$1{\cdot}00000_0$	—
0·2	1·22140	—	267_2	—	—	—	$1{\cdot}22140_3$	—
0·4	1·49182	$0{\cdot}49182_8$	326_8	59_6	—	-0_4	$1{\cdot}49182_4$	-0_4
0·6	1·82212	$0{\cdot}60072_0$	399_2	72_4	12_8	-0_5	$1{\cdot}82211_8$	-0_5
0·8	2·22554	$0{\cdot}73372_3$	487_5	88_3	15_9	-0_6	$2{\cdot}22554_1$	-1_0
1·0	2·71828	$0{\cdot}89617_1$	595_5	108_0	19_7	-0_8	$2{\cdot}71828_1$	-1_3
1·2	3·32012	$1{\cdot}09458_5$	727_3	131_8	23_8	-0_9	$3{\cdot}32011_7$	-1_9
1·4	4·05520	$1{\cdot}33693_1$	888_3	161_0	29_2	-1_2	$4{\cdot}05520_0$	-2_5
1·6	4·95303	$1{\cdot}63293_0$	1085_0	196_7	35_7	-1_4	$4{\cdot}95303_3$	-3_3
1·8	6·04965	$1{\cdot}99446_5$	1325_3	240_3	43_6	-1_7	$6{\cdot}04964_7$	-4_2
2·0	7·38906	$2{\cdot}43604_6$	1618_6	293_3	53_0	-2_1	$7{\cdot}38905_8$	-5_4
2·2	9·02501	$2{\cdot}97539_3$	1976_9	358_3	65_0	-2_6	$9{\cdot}02501_4$	-6_8

with appropriate initial conditions. In column 1 the values of e^x, as taken from five-figure tables, have been written down at interval 0·2. In column 2 the increments of y as calculated by Simpson's Rule are given, and in column 3 the D_n. The small figure in the sixth place is retained to reduce "rounding-off" error. The next two columns contain the two differences of D_n. Then the corrections calculated from (9 a) are given, and, finally, the corrected values of y. It will be seen that at $x=2{\cdot}2$ the total correction (shown in the last column) has amounted to about seven units in the fifth place, but that the corrected value is certainly correct to the fifth place.

Although this example shows the successful application of this correction we are not so much concerned with urging its use as with showing that the error of the process *can be determined* and that the accuracy of the results *can be specified*. When the error amounts to half a unit in the last place—as it does at the very beginning of our example—it is time to reduce the interval.

Finally, it may be noted that this method of test and correction can be employed in any continuous integration when Simpson's Rule is being used.

XCVI. *Resonance in Three-Electrode Valves.*

By E. W. B. GILL, *B.Sc., M.A., Fellow of Merton College, Oxford, and* R. H. DONALDSON, *M.A., University College, Oxford.**

IN a three-electrode valve in which the grid is kept at a positive potential V, and the anode at a slightly negative potential v with respect to the filament, the general motion of the electrons from the filament is to and fro from the space near the filament to the space near the anode until they are finally caught on the grid.

If the grid is fairly open, any electron may describe several such paths backwards and forwards so that it performs free oscillations between the filament and the anode before it is finally caught on the grid. The motion is not a simple harmonic motion, nor are the times of successive oscillations quite equal, as the electron may suffer slightly different deviations each time it passes the grid. But to a first approximation these times are the same, and will therefore be inversely proportional to $\sqrt{V}$.

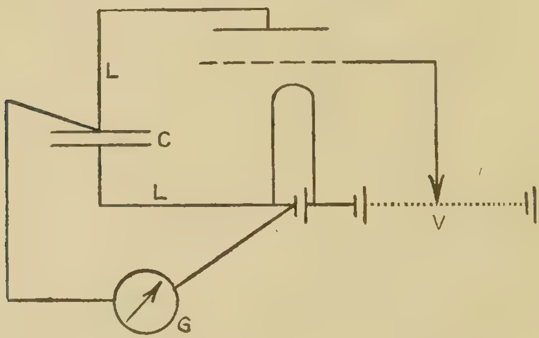
It is clear, therefore, that if an alternating potential whose periodic time is the same as that of the free oscillations of the electrons be superposed on the space between the filament and the anode a species of resonance should occur. The electrons leave the filament in a continuous stream, and those which leave at certain times have their amplitudes of oscillation increased while an equal number have their amplitudes decreased. Some of the electrons which have their amplitudes increased reach the anode.

* Communicated by Prof. J. S. Townsend, F.R.S.

Owing to the differences in the velocities of emission and direction of emission from the filament, and also owing to the different deviations when passing through the grid, there are inequalities in the amplitudes of the free oscillation of the different electrons, so that the number that reach the anode is a maximum when the period of the high frequency force is the same as that of the free oscillation.

In the experiments which we made the superposed high frequency potential was kept constant, both as regards amplitude and periodic time, and the time of the free oscillation of the electrons was varied by altering the grid potential V. The valve used was a transmitting valve of large dimensions, the anode being a cylinder of 4·7 cm.

Fig. 1.



diameter. The grid had only a few turns ; its diameter was 1 cm. approximately.

These large dimensions are desirable, as otherwise the periodicities needed to produce resonance for any reasonpale value of V would involve the use of inconveniently short waves.

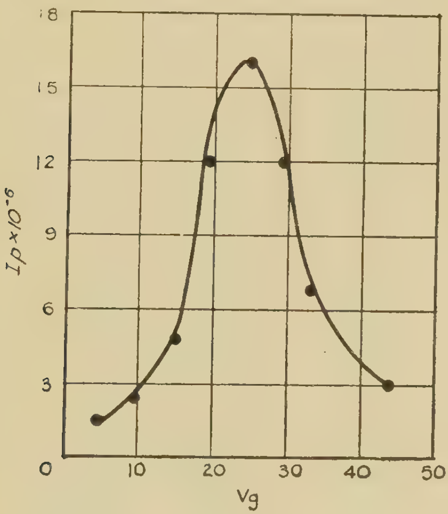
The grid was charged to a variable potential V as shown in fig. 1, and a loop of wire L was joined to the filament and anode. This loop was split at its middle point by the condenser C.

This condenser C was large enough to act as a short circuit for alternating currents, but any direct current reaching the anode had to return to the filament through the galvanometer G, the lead to which came in at one side of the condenser.

This loop L was placed near a short wave-generator, and as it is necessary that the alternating E.M.F. between the filament and the anode be very small the loop was not tuned to the generator.

The experiment consisted in noting the galvanometer deflexion for different values of the grid volts V. The curve (fig. 2) shows the result obtained with a wave-length of 10·01 metres and indicates a distinct resonance for V=24 volts approximately.

Fig. 2.



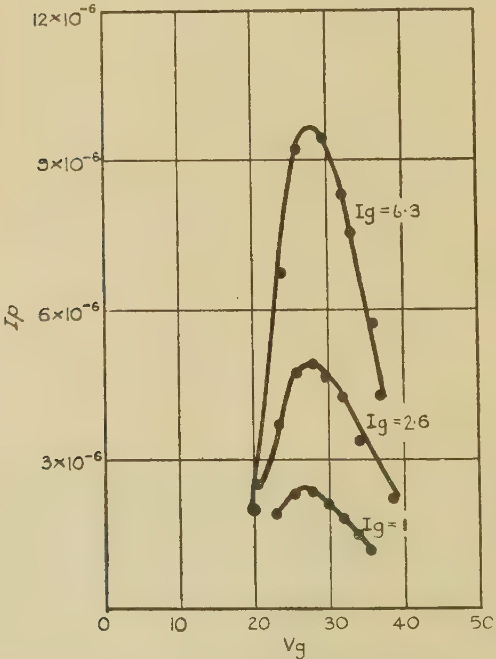
It is important, in view of recent theories, to be sure that the grid potential V at which resonance occurs is independent of the filament emission, and fig. 3 gives the results for oscillations of the wave-length 9·4 metres for three different emissions 1, 2·6, and 6·3 milliamps. respectively (as measured by the grid current). The resonance voltage in all three cases is practically the same, a slight increase for the highest emission being due to the larger space charges which alter the electric field and increase the time of transit of the electrons.

The time that the electrons take to move from the filament to anode is inversely proportional to $\sqrt{V}$, and if λ is the wave-length af the oscillation at which resonance occurs then λ^2V should be constant.

A series of experiments with different wave-lengths λ was made to test this, the grid potentials V being adjusted in each case to give resonance. The following table gives the results indicating that within errors of experiment λ^2V is constant. The chief error is due to the difficulty of estimating from the resonance curves the actual value of V for the peak.

λ (metres).	V.	λ^2V.
4·72	101·4	2260
5·4	80	2240
7·48	38·5	2170
10·01	23·4	2350

Fig. 3.



It should be possible to calculate the value of the constant by working out the time t that an electron takes to pass from the filament to the anode, but the expression for the time for the first part of the path from filament to grid involves the ratio of the filament diameter to grid diameter, and this was not known at all accurately. As an approximation, if the filament grid and anode are

taken as parallel planes, d_1 being distance from filament to grid and d_2 that from grid to anode, the time t (neglecting space charges) that the electron takes to pass from filament to anode is $(d_1+d_2)\sqrt{\frac{2m}{e\mathrm{V}}}$, which gives for this valve (measuring V in volts and taking e/m as 5.8×10^{17}) $t=7{\cdot}9/10^8\sqrt{\mathrm{V}}$.

Thus if λ be the wave-length whose half period is t,

$$\lambda^2\mathrm{V}=2247,$$

which is in close agreement with the experimental results.

One important result which we have mentioned is that the resonance is practically independent of the filament emission. Theories have recently been put forward to explain the Barkhausen-Kurz ultra short-wave oscillations based on the idea that the system of space charges in a valve has a normal mode of free oscillation in which the period is proportional to the inverse square root of the number of electrons between the electrodes at any instant *.

The results of the experiments shown by the curves (fig. 3) are not in agreement with this conclusion, as they show that resonance is obtained with oscillations of the same period, although the space charge changes in the proportion of 1 to 6·3.

In conclusion, we wish to thank Prof. J. S. Townsend, in whose laboratory this work was conducted.

XCVII. *Notices respecting New Books.*

Tables Annuelles Internationales de Constantes et Données Numeriques. Vol. VIII. 1ère Partie, Années 1927–28 (pub. 1931); Vol. VIII. 2me Partie, Années 1927–28 (pub. 1932); Vol. IX. Année 1929 (pub. 1931); Vol. IX. Table des Matières 1929 (pub. 1932). (The prices of these volumes are, Vol. VIII., both parts 500 f.; Vol. IX. 400 f.)

THESE tremendous productions comprise over 2700 pages for the two parts of Volume VIII. and over 1600 pages for Volume IX., and represent an enormous amount of very useful and careful abstracting of current scientific literature.

* Prof. A. Rostagni, Phil. Mag. p. 733 (March 1932).

As is well known, these Tables aim at giving in the main the outcome of new work involving numerical results, so that the volumes are literally packed with tables and curves covering recent work in diverse fields. Printed in French and English (with an occasional slip in the latter language), with the main headings in German and Italian as well, the range of subjects covered includes every branch of Physics, many tables and abstracts in Chemistry, Biology, Engineering, and Metallurgy. Altogether an astonishingly wide field is covered.

References are easily found, as the table of contents at the beginning of each volume is so easily handled.

It is hardly possible to criticize a production of this kind adversely, and it would be futile to search for small and unimportant errors; but one wonders whether such heavy volumes could not be split up into smaller parts or whether very much thinner and lighter paper could not be used, after the manner of a certain well-known encyclopædia—the aim being to lighten the sheer physical strain of dealing with such ponderous volumes.

It has been suggested also to the reviewer that many workers in individual fields would doubtless welcome the opportunity of being able to acquire reprints of certain sections of the tables separately.

Under suitable precautionary conditions facilities of this kind need not prejudice the successful issue of the work as a whole, while they would undoubtedly be of considerable help to individual workers in specialized fields.

It is curious to notice that the second part of Volume VIII., which dealt with papers published in 1927–28, appeared in 1932, while volume IX., dealing with papers published in 1929, was published in 1931.

If this apparent anomaly means that a new system is likely to mean in the future a still further reduction in the inevitable lag between the appearance of the original papers and the publication of the Tables Annuelles, then it is indeed a matter for still further congratulation.

The printing is clear and not difficult to read—altogether a most useful production.

British Association Mathematical Tables.—Vol. II. *Emden Functions.* Office of the British Association, Burlington House, London, W. 1. 1932.

THE already great debt that scientific investigators owe to the British Association Committee for the Calculation of Mathematical Tables is further increased by the issue of this book,

which they publish jointly with the International Astronomical Union. These tables are essential to anyone interested in the theory of stellar structure, and the need for a new issue of them had recently become acute, owing to the difficulty of obtaining copies of Emden's "Gaskugeln." The solutions of Emden's equation for nine values of the index n, ranging from 1 to 5 at intervals of half a unit, are given. Only solutions which have no singularity at the zero of the independent variable are included in this volume. In addition, certain associated functions, which are invariant with respect to the constant of homology, have been tabulated. These functions have played an important part in E. A. Milne's recent theory of stellar structure. He uses them to solve the "equations of fit" which determine the boundaries between zones in the stellar model characterized by different phases of matter.

Mr. J. C. P. Miller and Mr. D. H. Sadler have performed the actual computations, using a method due to Dr. J. R. Airey, and are to be congratulated on the successful completion of their laborious task.

The Fourier Integral and certain of its Applications. By NORBERT WIENER. (Cambridge University Press. 12*s.* 6*d.* net.)

THIS volume deals with a portion of Mathematical Analysis which has perhaps been rather neglected in England until quite recently. The author gave a series of lectures in Cambridge during the past academic year upon his own work in the subject, and has now used these lectures as the basis of his book. The publication of this work should render Professor Wiener's important results available and familiar to all mathematical analysts.

The book opens with an introductory section concerning results in Lebesque Integration theory, results which are needed in the course of the work. The first non-introductory chapter deals with the formal theory of the Fourier Transform and Plancherel's theorem. In Chapter II. the author sets out his general Tauberian theorem with its proof, and devotes the following chapter to individual Tauberian theorems and their relations with certain results in the theory of numbers. The final chapter is devoted to Generalized Harmonic Analysis and the theory of Almost Periodic Functions.

The work is well and clearly set out throughout the book, and the author nowhere allows himself to depart from an excellent economy of words.

Classical Theory of Electricity and Magnetism. By MAX ABRAHAM. Revised by RICHARD BECKER. Translated by JOHN DOUGALL, D.Sc. (Vol. V. of the 'Student's Physics Series,' published by Blackie. Cloth covers, 285 pp. $9'' \times 5''$. Price 15*s*.)

ABRAHAM'S 'Electricity and Magnetism' is too familiar to require fresh reviewing. The present book is a revision in which the interest of the physical content has been stressed rather than the pure development of the subject. A great increase in the number of diagrams has been made and an excellent series of examples, together with skeleton solutions where necessary, is included at the end. The result is a text-book suitable for examination purposes, if need be, as well as a volume to which reference can be made on points that arise in the course of either teaching or research.

The printing and general production are excellent.

Caractéristiques des systèmes différentiels et propagation des ondes. By TULLIO LEVI-CIVITA. [Pp. x+114.] (Paris: Librairie Alcan. Price 20 fr.)

THIS monograph, a translation from the Italian work 'Caratteristiche e propagazione ondosa' (Bologna, 1931) is written with Prof. Levi-Civita's customary precision and elegance. It expounds a subject on the border-line between pure mathematics and mathematical physics that is often neglected, inasmuch as it is often difficult to tempt the pure mathematician to interest himself in particular cases of theorems of wide generality, whilst the applied mathematician is often liable to be content with *ad hoc* methods of treatment. The present exposition, accordingly, may be regarded according to personal bias as the business of either or neither or both.

A brief sketch of the content serves to illustrate this point. The Cauchy-Kovalevsky theorem affirms the existence of regular solutions of a normal system of partial differential equations in the neighbourhood of given initial values. The equations of the small motion of a fluid or of an elastic solid, the equations of the electromagnetic field, as well as Schrödinger's equation, come under this theory. The condition that regular solutions exist in the neighbourhood of a given surface involves the non-vanishing of a certain determinant Ω. If, as in the case of a sound-pulse, there are discontinuities in some of the derivatives (here the first derivatives of the velocity-components), there must be surfaces at which regular solutions do not exist; these surfaces of discontinuity, or "characteristics" as they are called in this book, may therefore be found by equating Ω to zero. It should be mentioned that the "characteristics" of the customary Cauchy theory

of partial differential equations are referred to in this exposition as "bicharacteristics."

To quote an example: if the equation $\Omega=0$ be obtained for the classical hydrodynamical equations of a non-viscous fluid, it is found that Ω breaks up into two factors, one corresponding to the usual "dynamical surface condition" for a boundary surface, and the other to a surface of discontinuity propagated relative to the fluid with the velocity of a compressional wave. A further example of great interest is the proof of the impossibility of the propagation of a discontinuity in a viscous medium.

The not-very-mathematical reader will probably be overawed by the imposing generality of the problem proposed in the first few pages, so that he may, indeed, think it advisable to proceed no further. For this reason, a brief preliminary outline, perhaps a development of the preface, would be a help to mathematicians and physicists alike. Prof. Levi-Civita does, in fact, confine himself in the end to first- and second-order equations, and a somewhat more pedestrian exposition of the theory of normal systems might be preferred as an introduction.

The typography is very good, although the display of all proper names in large capitals is apt to be somewhat distracting. A certain number of misprints have been noted that should not have escaped a vigilant proof-reader. It remains only to express gratitude to Prof. Levi-Civita for the admirable manner in which he has carried out a real service to science.

Introductory Acoustics. By G. WALTER STEWART, Professor of Physics in the University of Iowa. (New York: Van Nostrand & Co. Price 2·75 dollars.)

THIS book can best be described as an introduction to the comparatively new phenomena met with in architectural acoustics intended for those who have had no training in the mathematical side of the subject. It may be found useful, therefore, by the professional man (an architect for example) who is now confronted with the problems discussed. But the pains which the author has taken to avoid the mathematical side of the subject make it quite useless as a text-book for a student beginning the thorough study of the subject. If only the trouble which has been taken to avoid an accurate mathematical statement in the first chapter had been devoted to giving a simple introduction to the very small amount of mathematics required the volume would have acquired a value which it does not possess in its present form.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

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END OF THE FIFTEENTH VOLUME.

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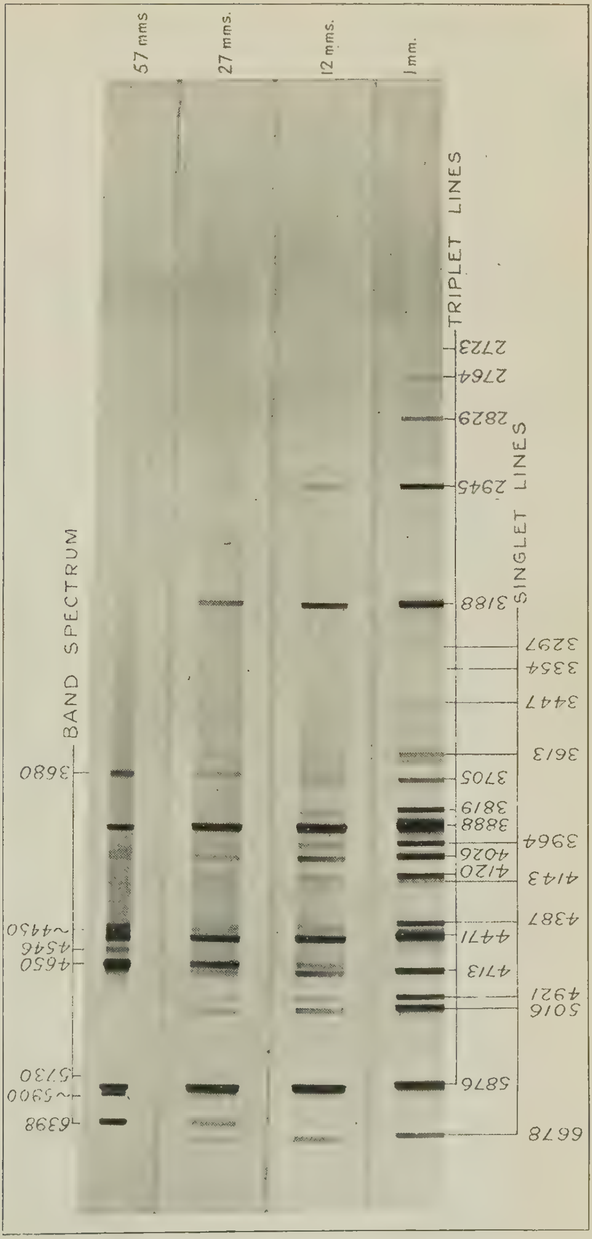
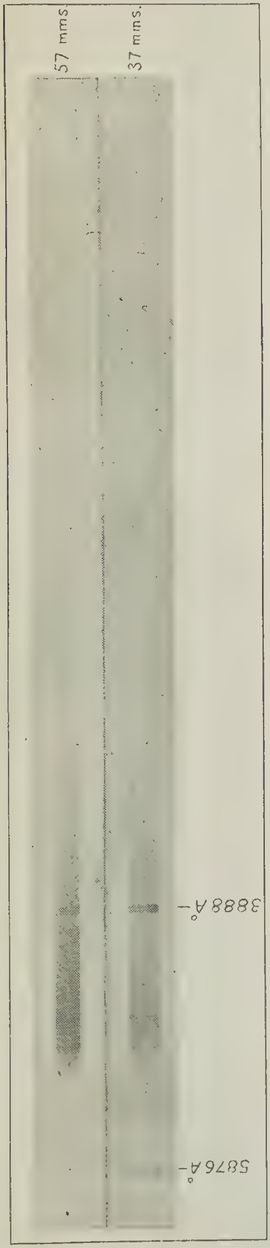
FIG. 1.



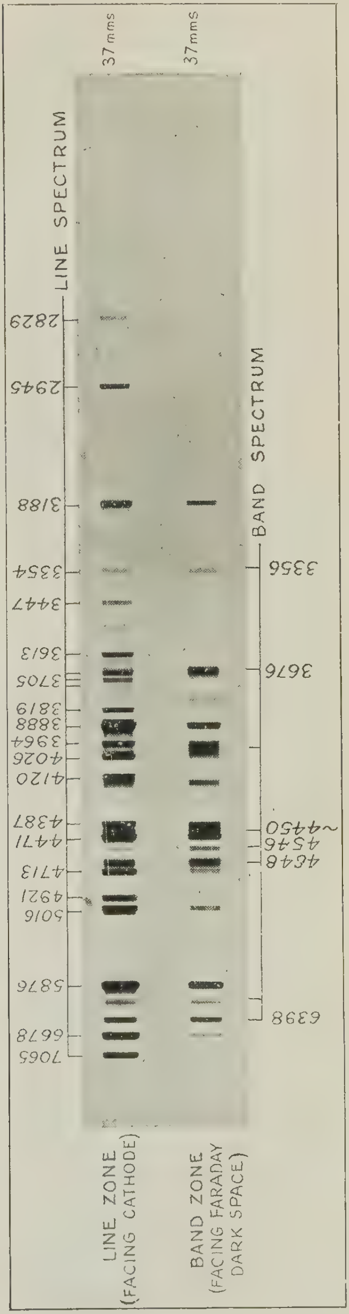
Spectra of the uniform column of a helium discharge.

Fig. 3.



Spectrum of the "green glow."

Fig. 4.



Spectrum of the cathode glow.